Chapter 3

Micromechanics of a Composite Lamina

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Contents of Chapter 3

3.1. Introduction

Macromechanics: The study of composite material behavior wherein the material is assumed *homogeneous* and the effects of the constituents are detected only as averaged apparent macroscopic properties of the composite material.

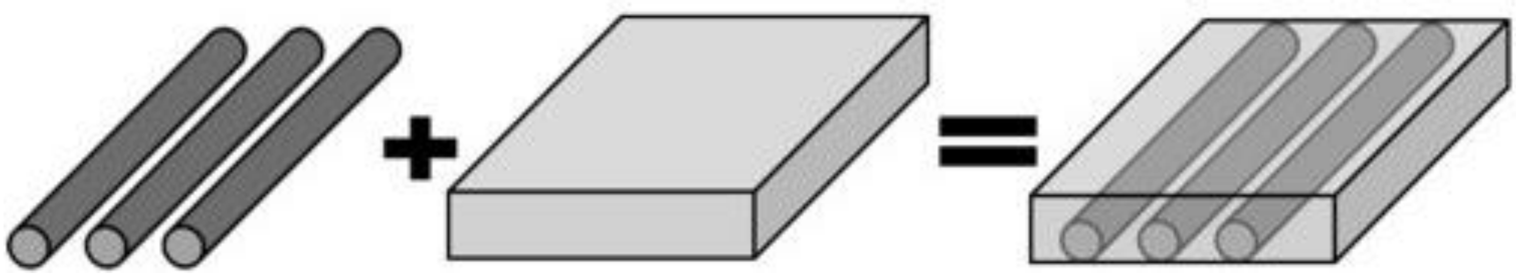
Micromechanics: The study of composite material behavior wherein the *interaction* of the constituent materials is examined on a *microscopic scale* to determine their effect on the properties of the composite material.

What are the constituent materials?

Micromechanics: constituents = fibers + matrix
Result = effective engineering properties

Macromechanics: constituents = lamina of different types and orientations
Result varies: effective engineering properties, ABD matrices

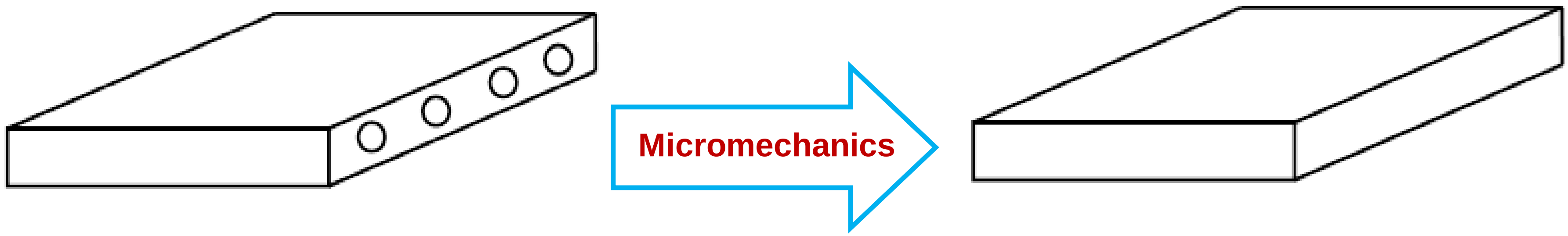
Micromechanics of Composite Materials



Reinforcement fiber	Matrix	Composite material
E_f	E_m	$E_c = ?$
X_f	X_m	$X_c = ?$

The basic question of micromechanics is: what is the relationship of the composite material properties to the properties of the constituents?

Micromechanics of a Lamina



(a) Heterogereous lamina (Nonhomogeneous lamina)

(b) Equivalent homogeneous anisotropic lamina

Micromechanical process

3.1. Introduction

Objective of Micromechanics

Determine the elastic moduli or stiffness or compliances of a composite material in terms of the elastic moduli of the constituent materials.

Basic Assumptions

- Lamina:

    – Initially stress-free, macroscopically homogeneous, linearly elastic, macroscopically orthotropic

- Fibers:

    – Homogeneous, regularly spaced, linearly elastic, perfectly aligned, isotropic, perfectly bonded

- Matrix:

    – Homogeneous, isotropic, linearly elastic, void-free

Notations

v	volume
V	volume fraction
ρ	density
w	weight
W	weigth fractions

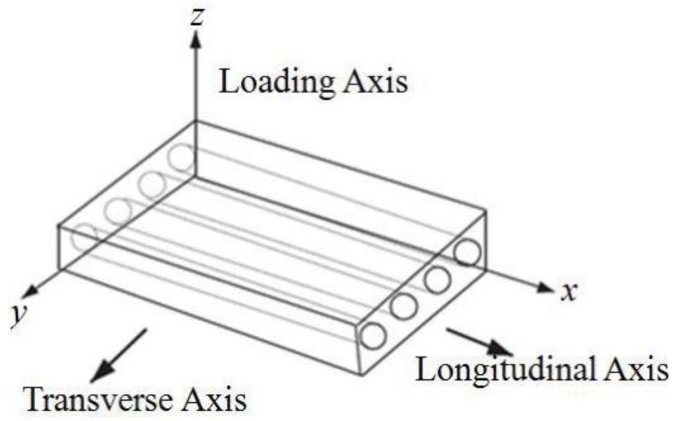
Terminology used in micromechanics

E_f, E_m – Young's modulus of fiber and matrix

G_f, G_m – Shear modulus of fiber and matrix

ν_f, ν_m – Poisson's ratio of fiber and matrix

V_f, V_m – Volume fraction of fiber and matrix



Subscript *f*, *m*, c refer to fiber, matrix, composite ply, respectively

Volume fractions

The volume of the composite material is equal to the sum of the volume of the fibers and the volume of the matrix. Therefore,

$$v_c = v_f + v_m$$

The volume fraction of the fiber and the matrix are defined as,

$$V_f = \frac{v_f}{v_c}, \text{ and } V_m = \frac{v_m}{v_c}$$

such that the sum of volume fractions is,

$$\boxed{V_f + V_m = 1}$$

Weight fractions (mass fractions)

The weight of the composite material is equal to the sum of the weight of the fibers and the weight of the matrix. Therefore,

$$w_f + w_m = w_c$$

The weight fractions (mass fractions) of the fiber and matrix are defined

$$W_f = \frac{w_f}{w_c}, \text{and} \quad W_m = \frac{w_m}{w_c}$$

such that the sum of weight fractions is

$$W_f + W_m = 1$$

Density

The density of composite material can be defined as the ratio of its weight to its volume and is expressed as,

$$\rho_c = \frac{w_c}{v_c} \Rightarrow w_c = \rho_c v_c \text{ or } v_c = \frac{w_c}{\rho_c}$$

But $v_c = v_f + v_m$

$$\Rightarrow \frac{w_c}{\rho_c} = \frac{w_f}{\rho_f} + \frac{w_m}{\rho_m} \Rightarrow \frac{1}{\rho_c} = \frac{1}{\rho_f}\left(\frac{w_f}{w_c}\right) + \frac{1}{\rho_m}\left(\frac{w_m}{w_c}\right)$$

By writing in terms of weight fractions

$$\frac{1}{\rho_c} = \frac{W_f}{\rho_f} + \frac{W_m}{\rho_m} \Rightarrow \boxed{\rho_c = \frac{1}{\left(\frac{W_f}{\rho_f} + \frac{W_m}{\rho_m}\right)} = \frac{1}{\sum_{i=1}^{n}\left(\frac{W_i}{\rho_i}\right)}}$$

Density

Moreover, the equation $w_f + w_m = w_c$, can be rewritten as

$$\rho_c v_c = \rho_f v_f + \rho_m v_m$$

$$\Rightarrow \quad \rho_c = \rho_f \left(\frac{v_f}{v_c} \right) + \rho_m \left(\frac{v_m}{v_c} \right)$$

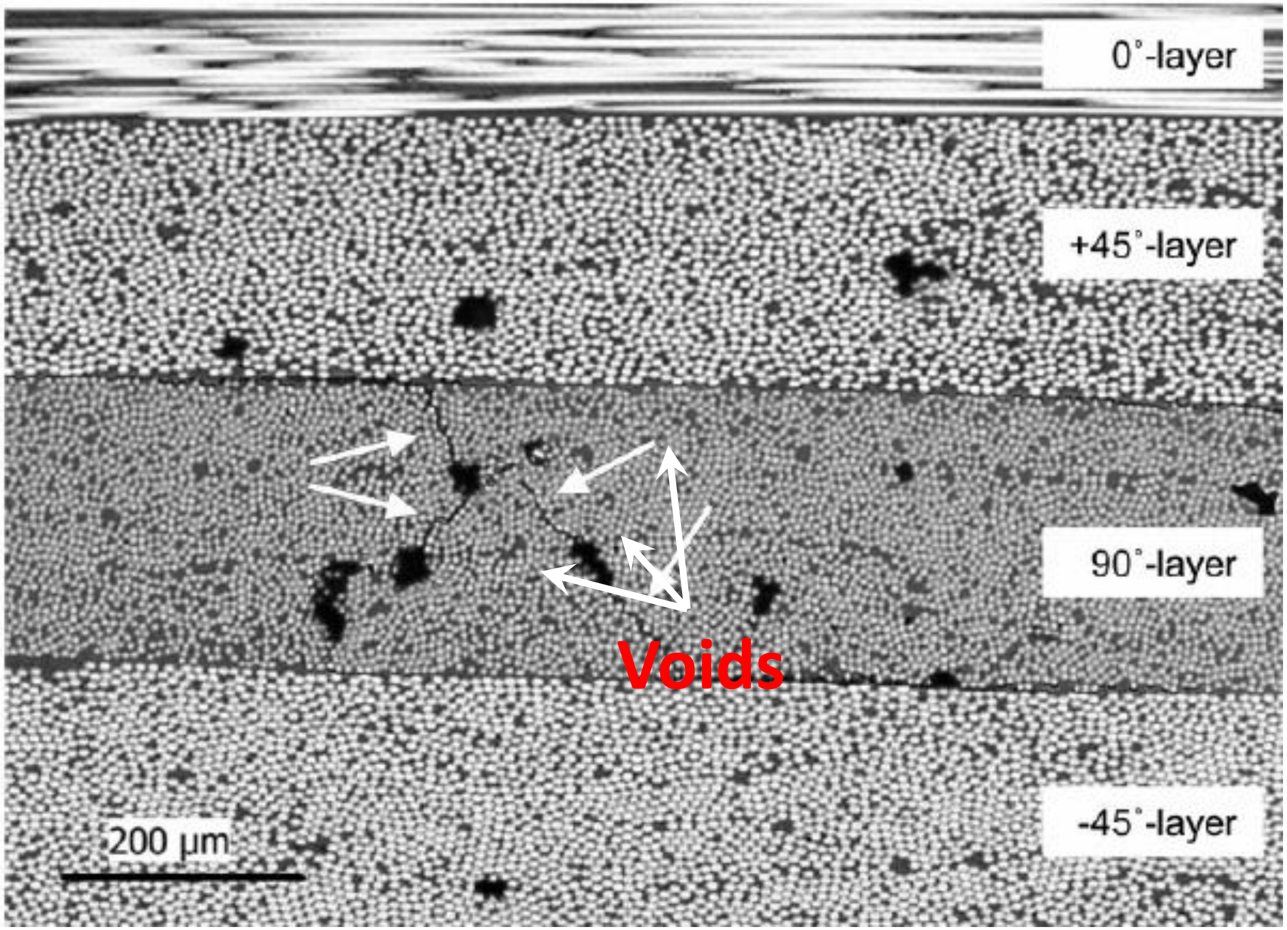
writing in terms of volume fractions, the density of the composite material is written as

$$\boxed{\rho_c = \rho_f V_f + \rho_m V_m}$$

In general

$$\boxed{\rho_c = \sum_{i=1}^{n} \rho_i V_i}$$

Void content



Micrographs of cross-section of a composite laminate with voids.

Void content

The volume of the composite $v_c = v_f + v_m + v_{void}$

$$v_c = \frac{w_c}{\rho_{ce}}, \text{and} \quad v_f + v_m = \frac{w_c}{\rho_{ct}}$$

where, ρ_{ct} - theoretical density of the composite material

ρ_{ce} - experimental density of the composite material

$$\frac{w_c}{\rho_{ce}} = \frac{w_c}{\rho_{ct}} + v_{void}, \text{ then } v_{void} = \frac{w_c}{\rho_{ce}}\left(\frac{\rho_{ct} - \rho_{ce}}{\rho_{ct}}\right)$$

The volume fraction of the voids $\boxed{V_{void} = \frac{v_{void}}{v_c} = \frac{\rho_{ct} - \rho_{ce}}{\rho_{ct}}}$

Table 3.1

Typical Properties of Fibers (SI System of Units)

Property	Units	Graphite	Glass	Aramid
Axial modulus	GPa	230	85	124
Transverse modulus	GPa	22	85	8
Axial Poisson's ratio	—	0.30	0.20	0.36
Transverse Poisson's ratio	—	0.35	0.20	0.37
Axial shear modulus	GPa	22	35.42	3
Axial coefficient of thermal expansion	µm/m/°C	–1.3	5	–5.0
Transverse coefficient of thermal expansion	µm/m/°C	7.0	5	4.1
Axial tensile strength	MPa	2067	1550	1379
Axial compressive strength	MPa	1999	1550	276
Transverse tensile strength	MPa	77	1550	7
Transverse compressive strength	MPa	42	1550	7
Shear strength	MPa	36	35	21
Specific gravity	—	1.8	2.5	1.4

Table 3.2

Typical Properties of Matrices (SI System of Units)

Property	Units	Epoxy	Aluminum	Polyamide
Axial modulus	GPa	3.4	71	3.5
Transverse modulus	GPa	3.4	71	3.5
Axial Poisson's ratio	—	0.30	0.30	0.35
Transverse Poisson's ratio	—	0.30	0.30	0.35
Axial shear modulus	GPa	1.308	27	1.3
Coefficient of thermal expansion	µm/m/°C	63	23	90
Coefficient of moisture expansion	m/m/kg/kg	0.33	0.00	0.33
Axial tensile strength	MPa	72	276	54
Axial compressive strength	MPa	102	276	108
Transverse tensile strength	MPa	72	276	54
Transverse compressive strength	MPa	102	276	108
Shear strength	MPa	34	138	54
Specific gravity	—	1.2	2.7	1.2

Example 3.1

A Glass/Epoxy lamina consists of a 70% fiber volume fraction. Use properties of glass and epoxy from Tables 3.1 and 3.2 (A.K. Kaw. Mechanics of Composite Materials. 2nd edition, Taylor & Francis, 2006), respectively to determine the

a) density of lamina
b) mass fractions of the glass and epoxy
c) volume of composite lamina if the mass of the lamina is 4 kg.
d) volume and mass of glass and epoxy in part (c).

Solution

a) Density of lamina $\rho_f = 2500\,kg/m^3$, and $\rho_m = 1200\,kg/m^3$

$$\rho_c = (2500)(0.7) + (1200)(0.3) = 2110\,kg/m^3$$

b) Mass (weight) fractions of the glass and epoxy

$$W_f = \frac{2500}{2110} \times 0.7 = 0.8294, \text{ and } \quad W_m = \frac{1200}{2110} \times 0.3 = 0.1706$$

$$W_c = W_f + W_m = 0.8294 + 0.1706 = 1.000$$

c) Volume of composite lamina if the mass of the lamina is 4 kg

$$v_c = \frac{w_c}{\rho_c} = \frac{4}{2110} = 1.896 \times 10^{-3} m^3$$

d) Volume and mass of glass and epoxy in part (c)

$$v_f = V_f v_c = (0.7)(1.896 \times 10^{-3}) = 1.327 \times 10^{-3} m^3$$

$$v_m = V_m v_c = (0.3)(0.1896 \times 10^{-3}) = 0.5688 \times 10^{-3} m^3$$

$$w_f = \rho_f v_f = (2500)(1.327 \times 10^{-3}) = 3.318\ kg$$

$$w_m = \rho_m v_m = (1200)(0.5688 \times 10^{-3}) = 0.682 \text{ kg}$$

Example 3.2: A graphite/epoxy cuboid specimen with voids has dimensions $a \times b \times c$ and its mass is M_c. After putting it in a mixture of sulphuric acid and hydrogen peroxide, the remaining graphite fibers have a mass M_f. The densities of graphite and epoxy are ρ_f and ρ_m, respectively. Find the volume fraction of the voids in terms of $a, b, c, M_f, M_c, \rho_f$ and ρ_m.

Solution

The volume of the composite: $v_c = v_f + v_m + v_{void}$

From the definition of density: $v_f = \dfrac{M_f}{\rho_f}$, $v_m = \dfrac{M_c - M_f}{\rho_m}$, and $v_c = abc$

Substituting

$$\Rightarrow abc = \frac{M_f}{\rho_f} + \frac{M_c - M_f}{\rho_m} + v_{void}$$

The volume fraction of voids is

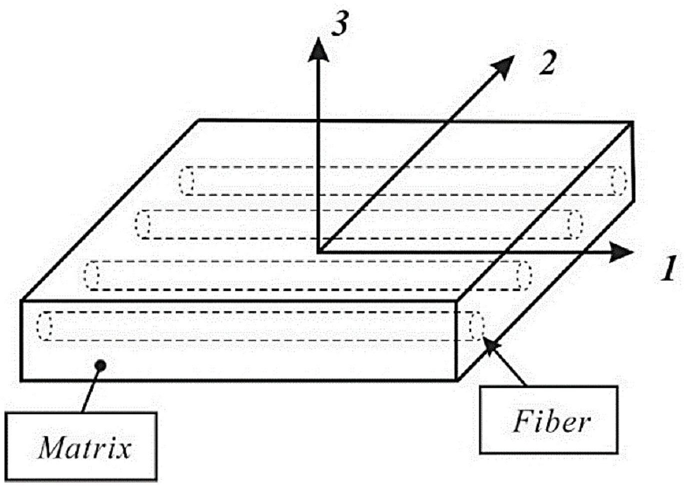
$$V_{void} = \frac{v_{void}}{abc} = 1 - \frac{1}{abc}\left[\frac{M_f}{\rho_f} + \frac{M_c - M_f}{\rho_m}\right]$$

3.3. Elastic moduli of a Unidirectional Lamina

The objective of all micromechanics approaches is to determine the elastic moduli or stiffnesses or compliances of a composite material in terms of the elastic moduli of the constituent materials:

There are four elastic moduli of a unidirectional lamina:

- Longitudinal Young's modulus, E_1
- Transverse Young's modulus, E_2
- Major Poission's ratio, ν_{12}
- In-plane shear modulus, G_{12}



Approaches to determine elastic moduli:

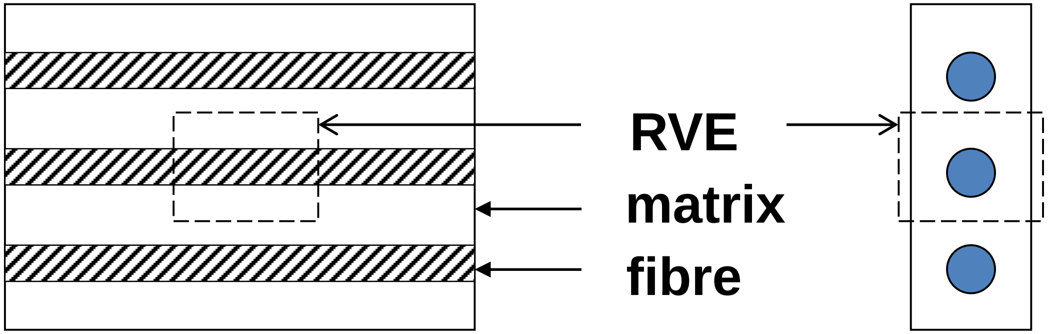
1. Mechanics of Materials Approach
2. Semi-Empirical Models: Halphin-Tsai Equation

1. Mechanics of materials approach

From a unidirectional lamina, take a representative volume element (RVE) that consists of the fiber surrounded by the matrix.

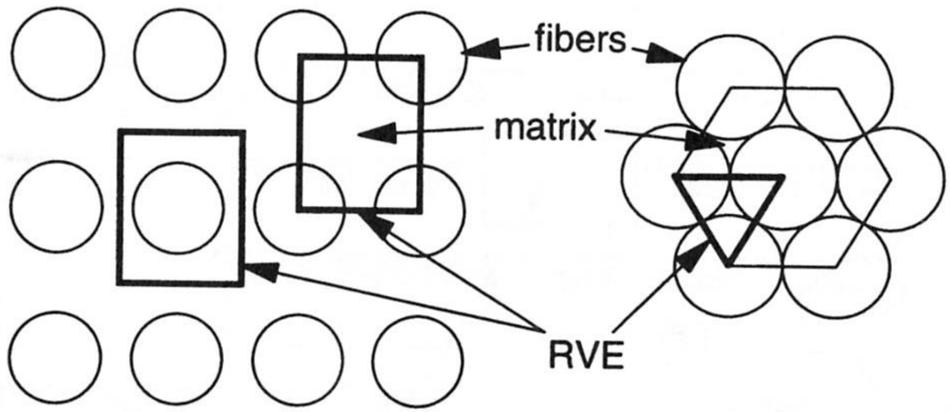
Representative Volume Element (RVE)

This is the smallest ply region over which the stresses and strains behave in a macroscopically homogeneous behavior.



Microscopically, RVE is of a heterogeneous behavior.

Generally, single force is considered in the RVE.

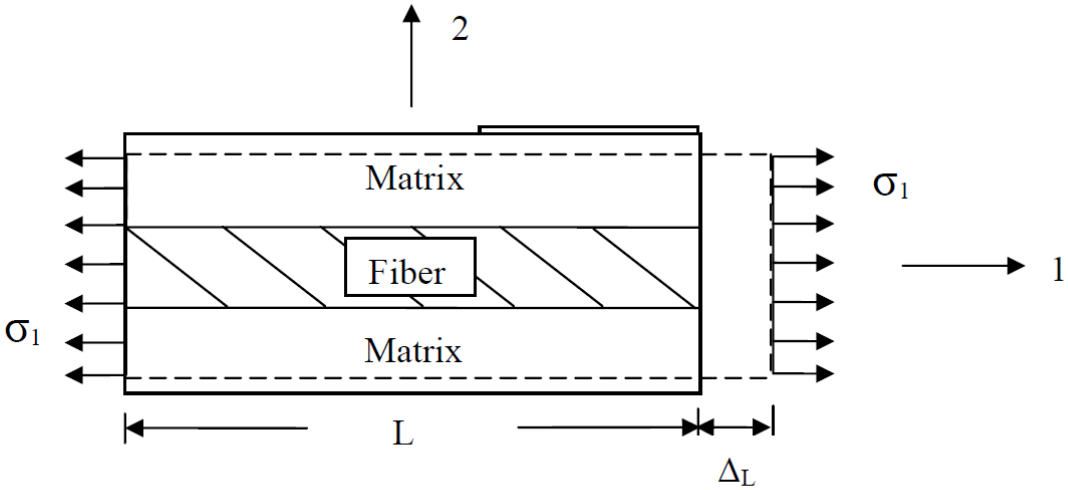


Assumptions are made in the mechanics of materials approach

- The bond between fibers and matrix is perfect.
- The elastic moduli, diameters, and space between fibers are uniform.
- The fibers are continuous and parallel.
- The fiber and matrix follow Hooke's law (linearly elastic).
- The fibers possess uniform strength.
- The composites is free of voids.

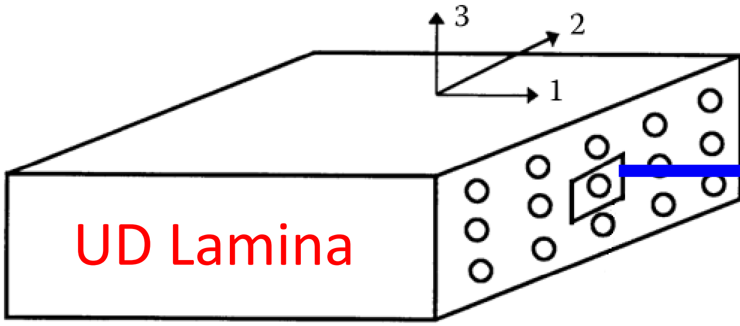
Most Prominent Assumptions

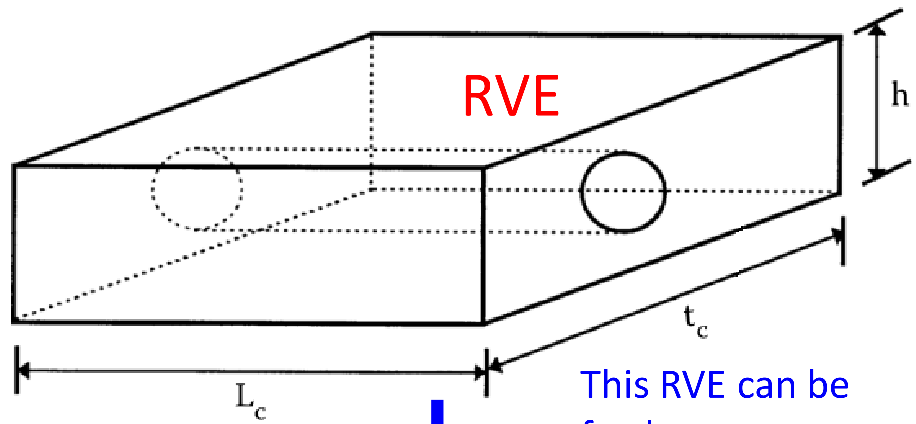
- In fiber direction, strains in fibers and matrix are equal ($\varepsilon_c = \varepsilon_f = \varepsilon_m$).
- Normal to fiber direction, stresses in matrix and fiber are equal ($\sigma_{c2} = \sigma_{f2} = \sigma_{m2}$).



Representative Volume Element (RVE)

RVE of a unidirectional lamina.



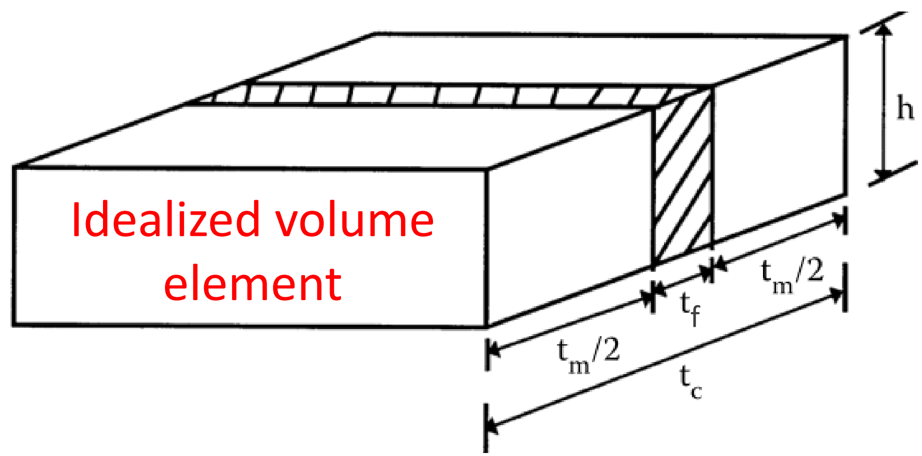


This RVE can be further represented as rectangular blocks.

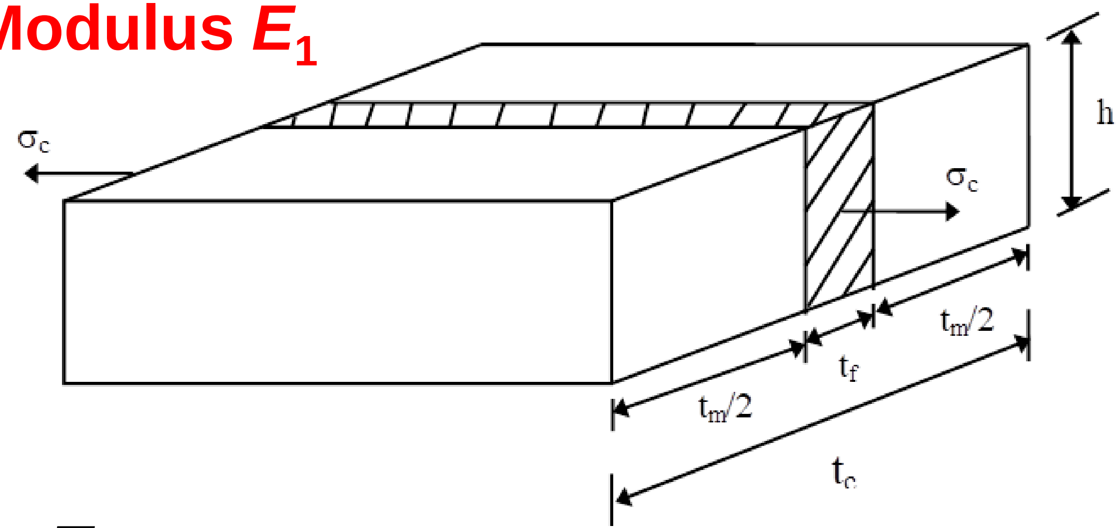
$$A_f = t_f h \, . \, A_m = t_m h, \; A_c = t_c h.$$

$$V_f = \frac{A_f}{A_c} = \frac{t_f}{t_c},$$

$$V_m = \frac{A_m}{A_c} = \frac{t_m}{t_c} = 1 - V_f.$$



Longitudinal Young's Modulus E_1



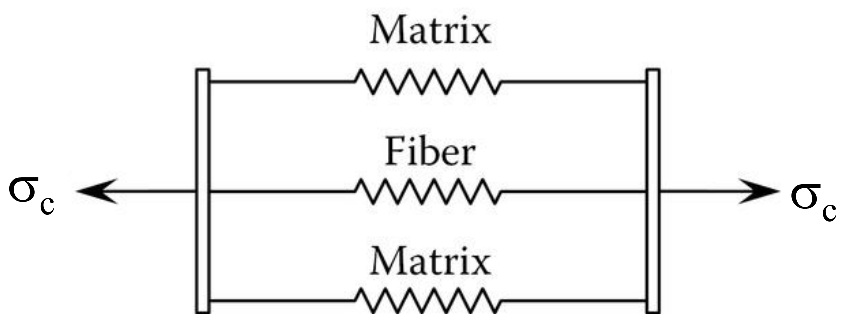
Total force is shared by fiber and matrix

$$F_c = F_f + F_m$$

$$F_c = \sigma_c A_c, \qquad \sigma_c = E_1 \varepsilon_c,$$

$$F_f = \sigma_f A_f, \quad \text{and} \quad \sigma_f = E_f \varepsilon_f$$

$$F_m = \sigma_m A_m \qquad \sigma_m = E_m \varepsilon_m$$

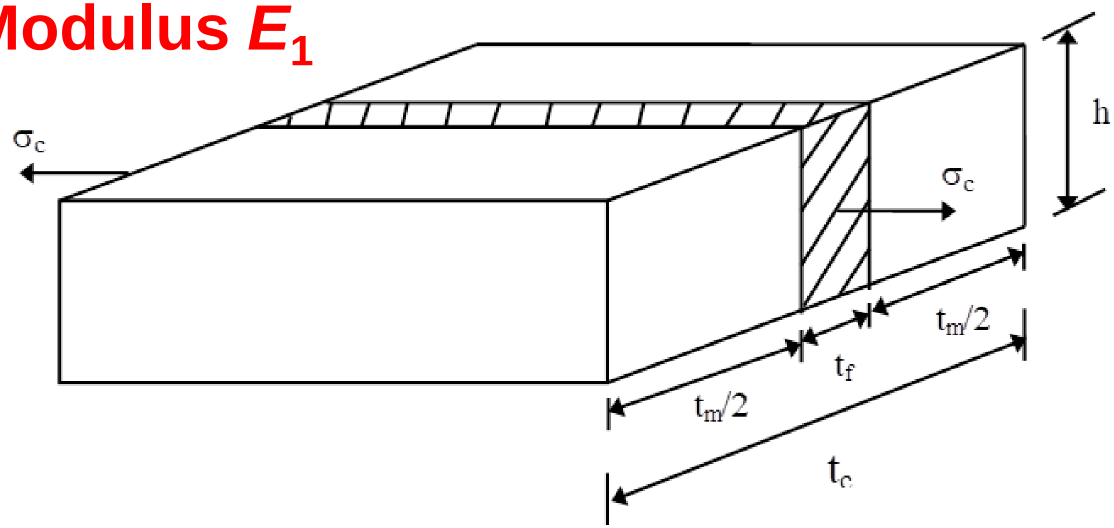


$\sigma_{c,f,m}$ = stress in composite, fiber, and matrix, respectively
$A_{c,f,m}$ = area of composite, fiber, and matrix, respectively

3.3. Elastic moduli of a Unidirectional Lamina

Longitudinal Young's Modulus E_1

$$F_c = F_f + F_m$$



$$E_1 \varepsilon_c A_c = E_f \varepsilon_f A_f + E_m \varepsilon_m A_m$$

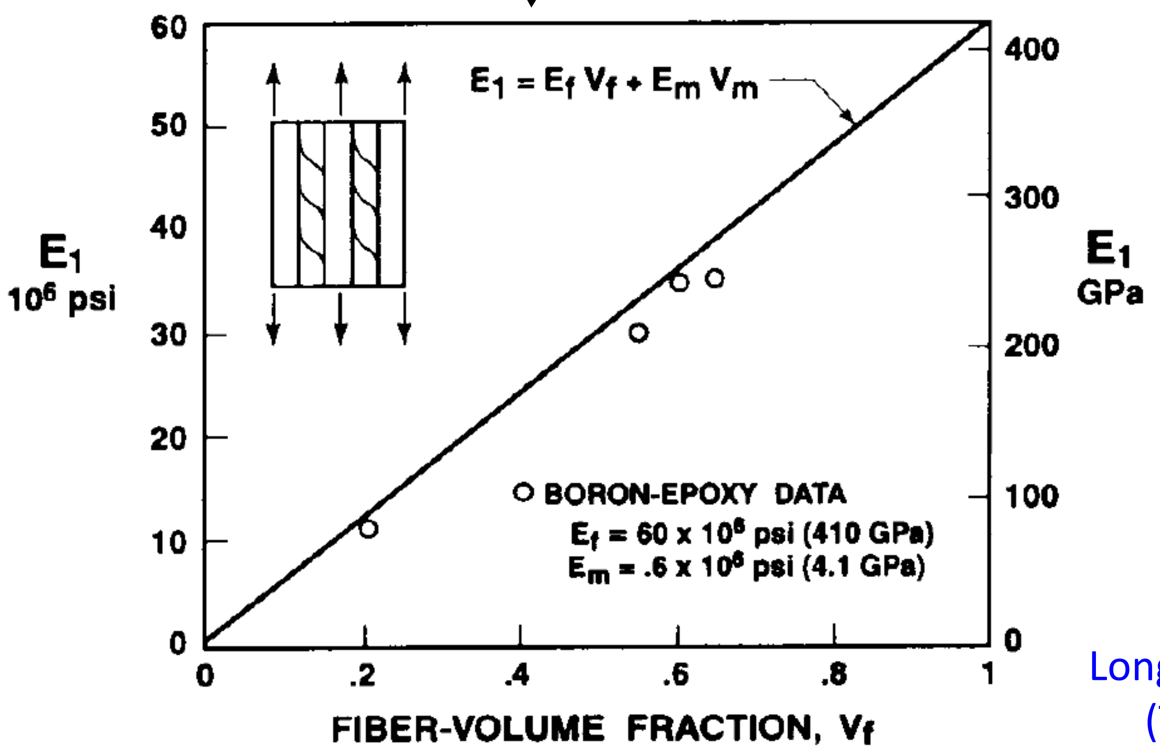
If $(\varepsilon_c = \varepsilon_f = \varepsilon_m)$, then:

Rule of Mixtures

$$E_1 = E_f \frac{A_f}{A_c} + E_m \frac{A_m}{A_c} \longrightarrow \boxed{E_1 = E_f V_f + E_m V_m}$$

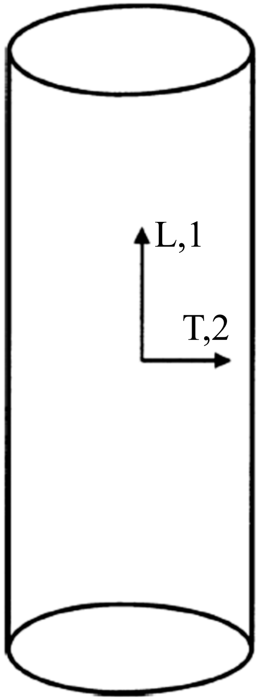
Rule of Mixtures

$$\boxed{E_1 = E_f V_f + E_m V_m}$$



For transversely isotropic fiber

$$\boxed{E_1 = E_{1f} V_f + E_m V_m}$$



Longitudinal (L,1) and transverse direction (T,2) in a transversely isotropic fiber.

3.3. Elastic moduli of a Unidirectional Lamina

Longitudinal Young's Modulus E_1

$$E_1 = E_f V_f + E_m V_m$$

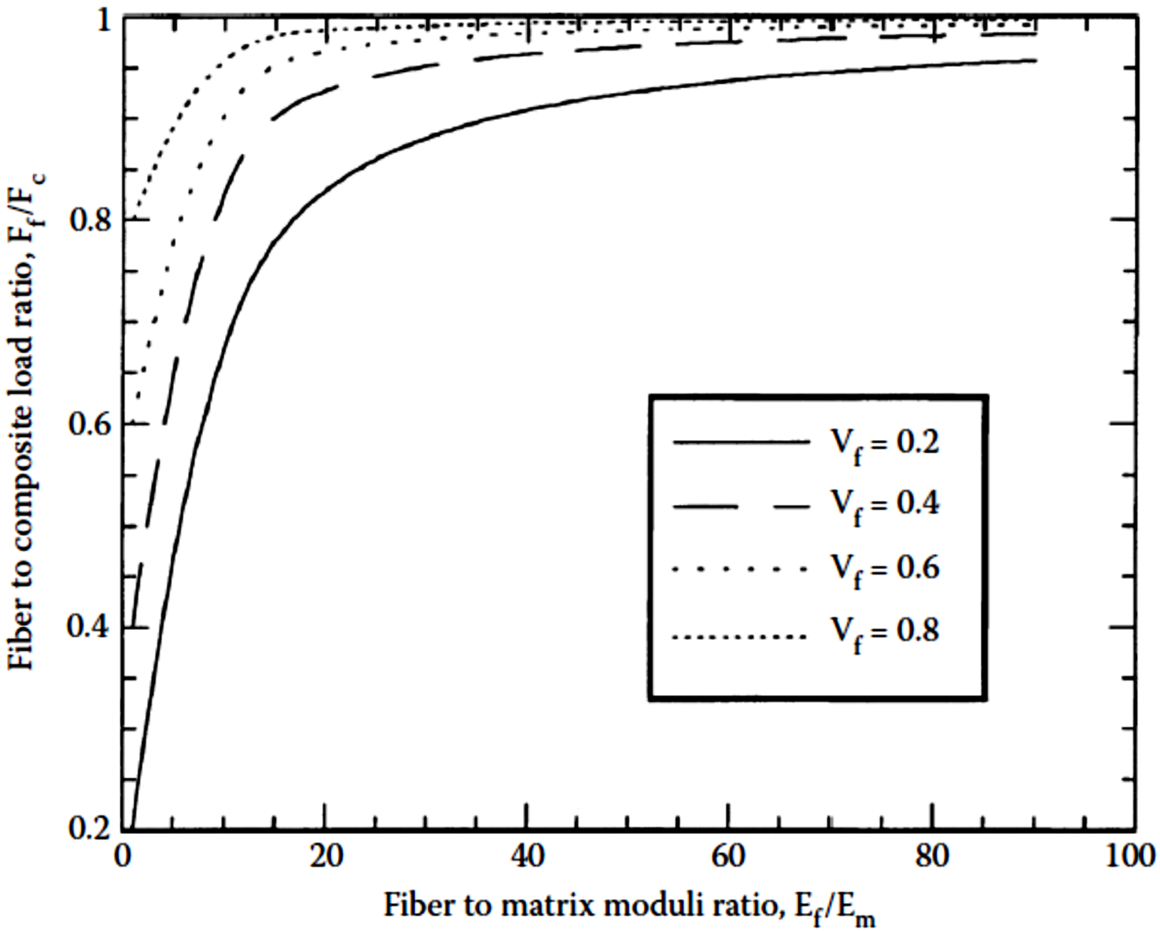
$$F_c = \sigma_c A_c, \qquad \sigma_c = E_1 \varepsilon_c,$$

$$F_f = \sigma_f A_f, \qquad \sigma_f = E_f \varepsilon_f$$

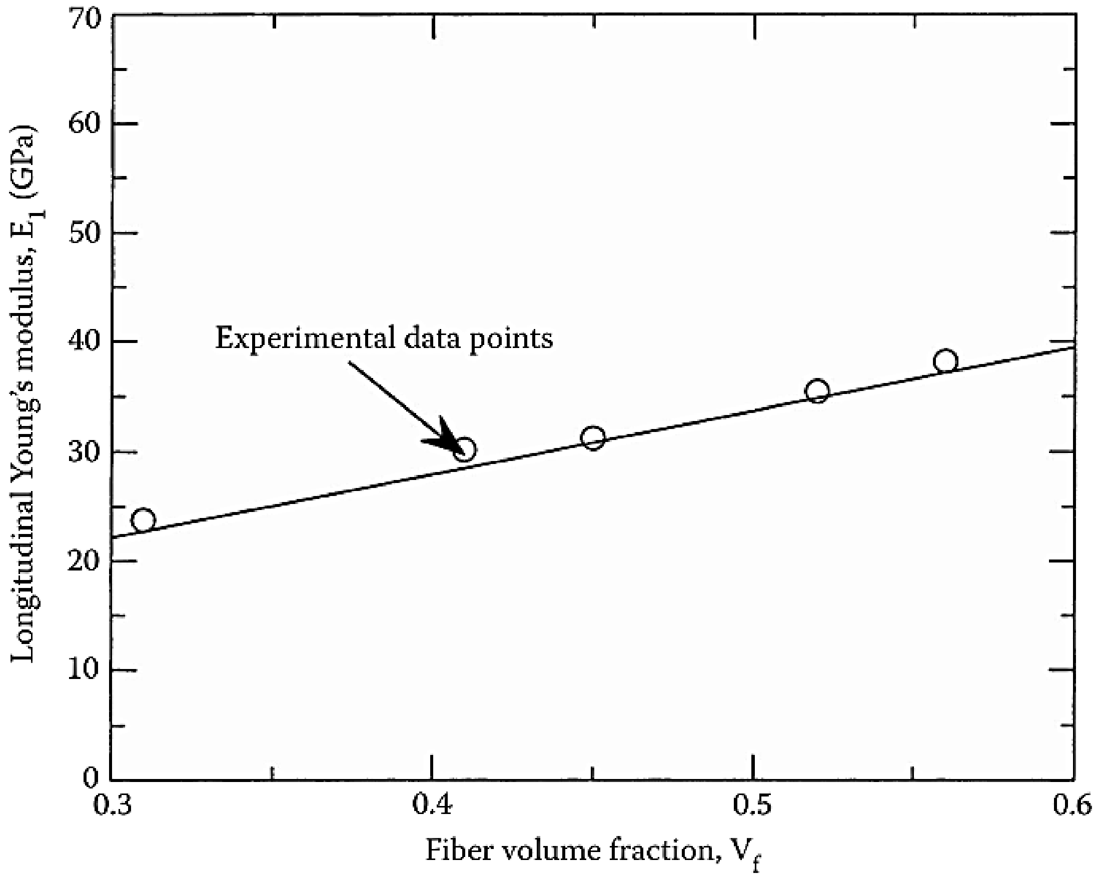
$$F_m = \sigma_m A_m \qquad \sigma_m = E_m \varepsilon_m$$

$$\longrightarrow \frac{F_f}{F_c} = \frac{E_f}{E_1} V_f$$

The ratio of the load taken by the fibers to the load taken by the composite is a measure of the load shared by the fibers.

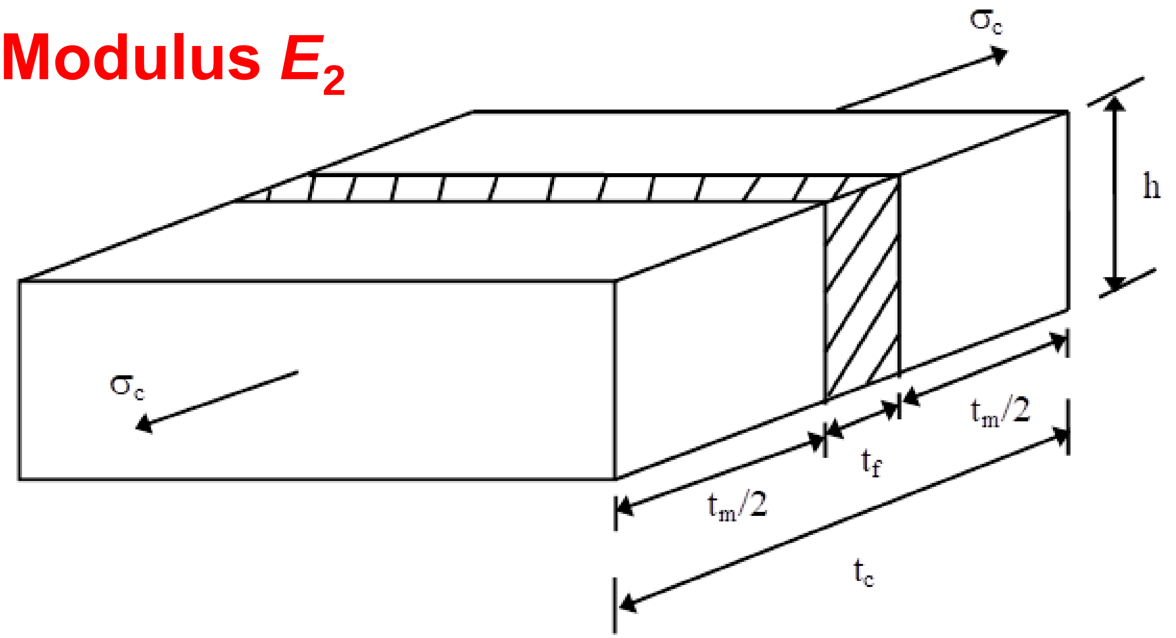


Longitudinal Young's Modulus E_1



Longitudinal Young's modulus as function of fiber volume fraction and comparison with experimental data for a typical glass/polyester lamina. Experimental data points reproduced with permission of ASM International.

Transverse Young's Modulus E_2



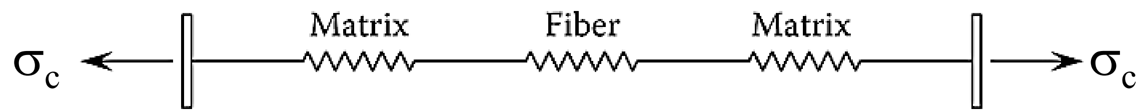
$$\sigma_c = \sigma_f = \sigma_m$$

$$\Delta_c = \Delta_f + \Delta_m$$

Δ is transverse extension

$$\Delta_c = t_c \, \varepsilon_c \,, \quad \varepsilon_c = \frac{\sigma_c}{E_2} \,,$$

$$\Delta_f = t_f \varepsilon_f \,, \quad \varepsilon_f = \frac{\sigma_f}{E_f} \,,$$



$$\Delta_m = t_m \varepsilon_m \,, \quad \varepsilon_m = \frac{\sigma_m}{E_m}$$

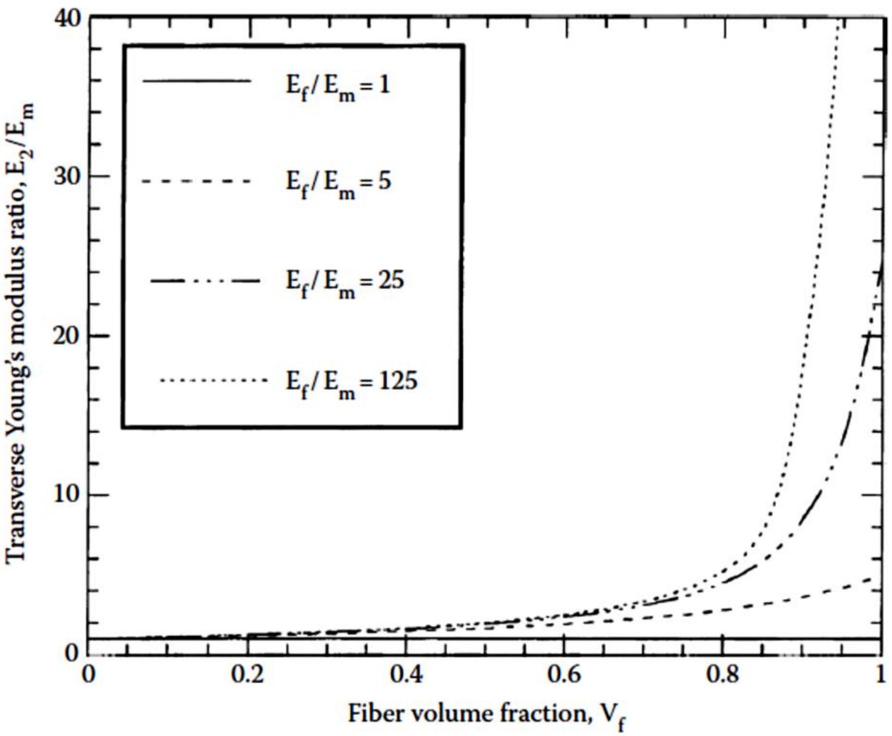
$$\rightarrow \frac{1}{E_2} = \frac{1}{E_f}\frac{t_f}{t_c} + \frac{1}{E_m}\frac{t_m}{t_c} \,,$$

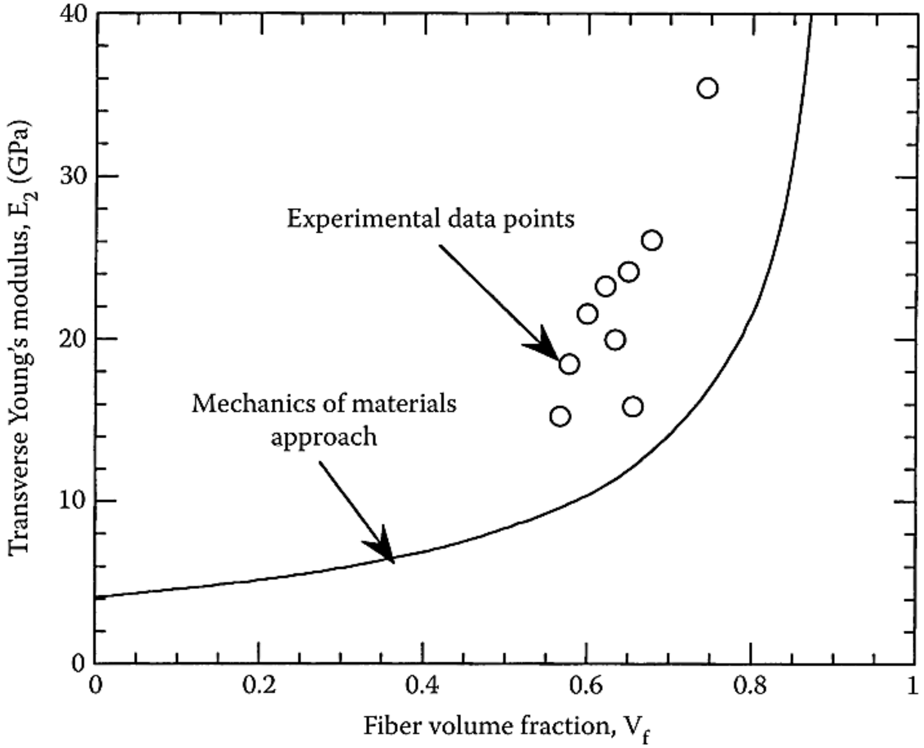
$$\boxed{\frac{1}{E_2} = \frac{V_f}{E_f} + \frac{V_m}{E_m}}$$

For transversely isotropic fiber

$$\boxed{\frac{1}{E_2} = \frac{V_f}{E_{2f}} + \frac{V_m}{E_m}}$$

Transverse Young's Modulus E_2



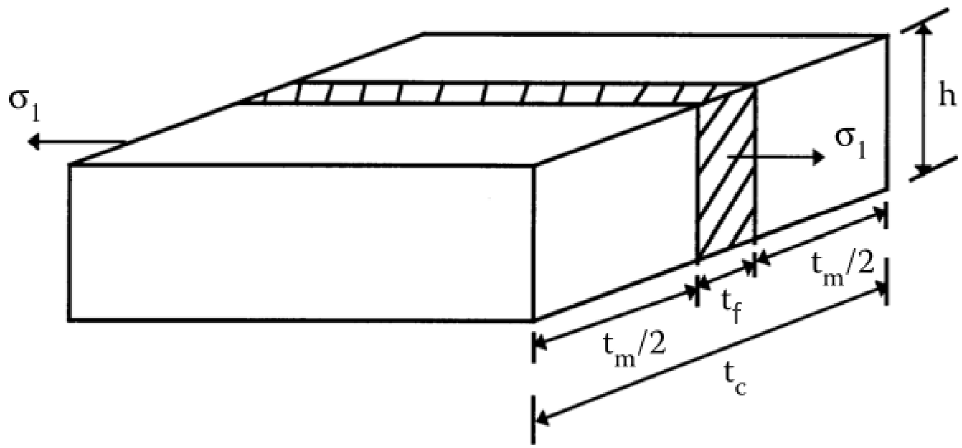


Transverse Young's modulus as a function of fiber volume fraction for constant fiber to matrix moduli ratio, E_f/E_m.

Theoretical values of transverse Young's modulus as a function of fiber volume fraction for a Boron/Epoxy unidirectional lamina (E_f = 414 GPa, ν_f = 0.2, E_m = 4.14 GPa, ν_m = 0.35) and comparison with experimental values.

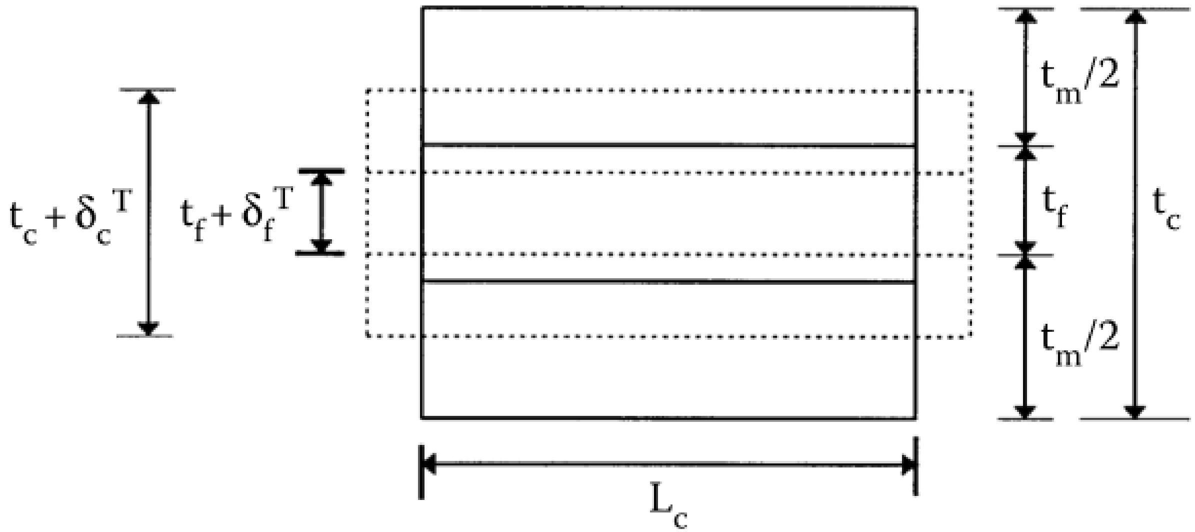
Major Poisson's Ratio, ν_{12}:

$$\nu_{12} = -\varepsilon^T / \varepsilon^L$$



The major Poisson's ratio is defined as the negative of the ratio of the normal strain in the transverse direction to the normal strain in the longitudinal direction, when a normal load is applied in the longitudinal direction.

A longitudinal stress applied to a representative volume element to calculate Poisson's ratio of unidirectional lamina



Major Poisson's Ratio, ν_{12} and Minor Poisson's Ratio, ν_{21}

The deformations in the transverse direction of the composite δ_c^T is the sum of the transverse deformations of the fiber δ_f^T and the matrix δ_m^T as

$$\delta_c^T = \delta_f^T + \delta_m^T$$

Using the definition of normal strains,

$$\varepsilon_f^T = \frac{\delta_f^T}{t_f}, \quad \varepsilon_m^T = \frac{\delta_m^T}{t_m}, \text{and} \quad \varepsilon_c^T = \frac{\delta_c^T}{t_c}$$

where $\varepsilon_{c,f,m}^T$ are transverse strains in composite, fiber, and matrix, respectively.

Substituting, $$t_c \varepsilon_c^T = t_f \varepsilon_f^T + t_m \varepsilon_m^T$$

Major Poisson's Ratio, ν_{12} and Minor Poisson's Ratio, ν_{21}

The Poisson's ratios for the fiber, matrix, and composite are

$$\nu_f = -\frac{\varepsilon_f^T}{\varepsilon_f^L}, \qquad \nu_m = -\frac{\varepsilon_m^T}{\varepsilon_m^L}, \text{and} \qquad \nu_{12} = -\frac{\varepsilon_c^T}{\varepsilon_c^L}$$

Substituting $-t_c \nu_{12} \varepsilon_c^L = -t_f \nu_f \varepsilon_f^L - t_m \nu_m \varepsilon_m^L$

However, the strains in the composite, fiber, and matrix are assumed to be the equal in the longitudinal direction, therefore

If $\varepsilon_c^L = \varepsilon_f^L = \varepsilon_m^L$, then: $t_c \nu_{12} = t_f \nu_f + t_m \nu_m \rightarrow \nu_{12} = \nu_f \dfrac{t_f}{t_c} + \nu_m \dfrac{t_m}{t_c}$

$$\rightarrow \boxed{\nu_{12} = \nu_f V_f + \nu_m V_m} \quad \text{Or} \quad \boxed{\nu_{12} = \nu_{12f} V_f + \nu_m V_m}$$

The minor Poisson's ratio is $\boxed{\nu_{21} = \nu_{12} \dfrac{E_2}{E_1}}$

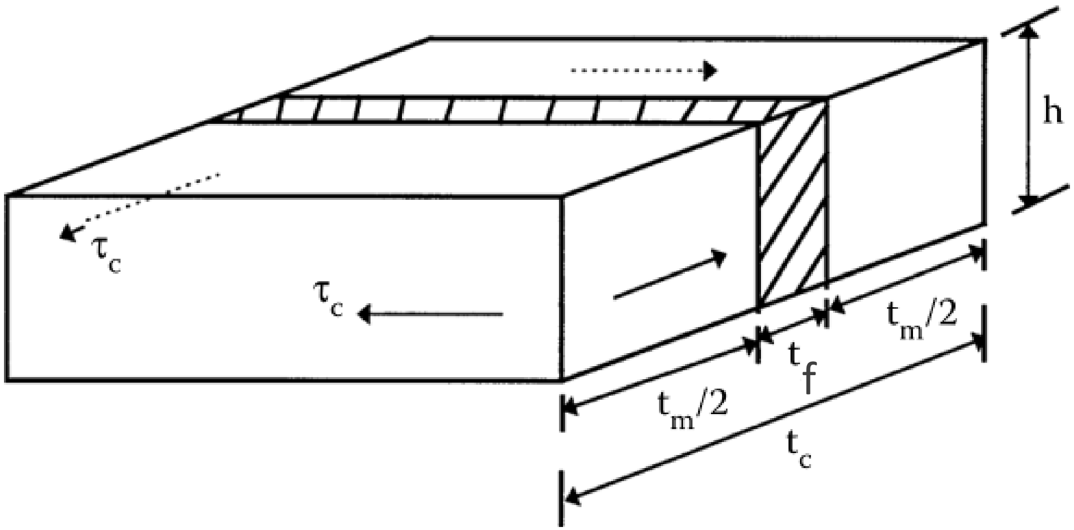
For transversely isotropic fiber

3.3. Elastic moduli of a Unidirectional Lamina

In-Plane Shear Modulus, G_{12}

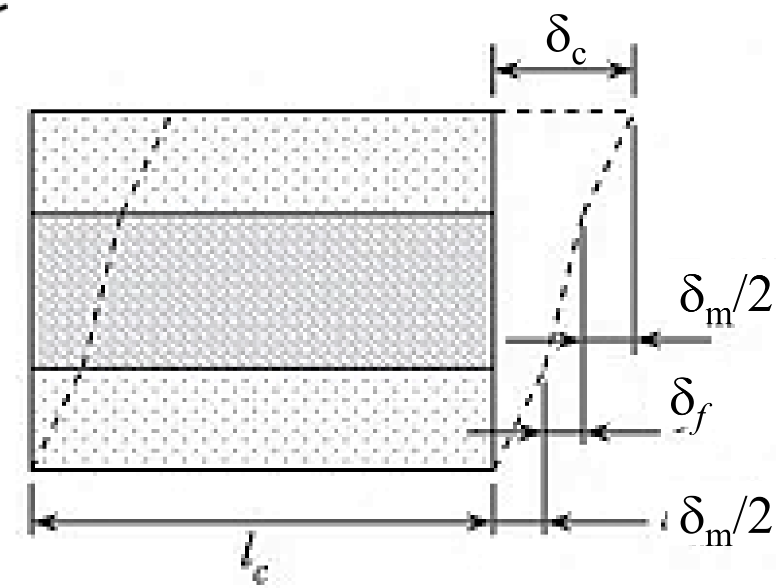
An in-plane shear stress applied to a representative volume element for finding in-plane shear modulus of a unidirectional lamina

Apply a pure shear stress τ_c to a lamina



Shear deformations

$$\delta_c = \delta_f + \delta_m$$



3.3. Elastic moduli of a Unidirectional Lamina

In-Plane Shear Modulus, G_{12}

The resulting shear deformations of the composite δ_c the fiber δ_f, and the matrix δ_m are related by

$$\delta_c = \delta_f + \delta_m$$

From the definition of shear strains,

$$\delta_c = \gamma_c t_c, \quad \delta_f = \gamma_f t_f, \text{and} \quad \delta_m = \gamma_m t_m$$

$\gamma_{c,f,m}$ = shearing strains in the composite, fiber, and matrix,
$t_{c,f,m}$ = thickness of the composite, fiber, and matrix, respectively.

From Hooke's law for the fiber, the matrix, and the composite,

$$\gamma_c = \frac{\tau_c}{G_{12}}, \quad \gamma_f = \frac{\tau_f}{G_f}, \text{ and} \quad \gamma_m = \frac{\tau_m}{G_m}$$

where $G_{12,f,m}$ = shear moduli of composite, fiber, and matrix.

In-Plane Shear Modulus, G_{12}

From equations above, we have

$$\frac{\tau_c}{G_{12}} t_c = \frac{\tau_f}{G_f} t_f + \frac{\tau_m}{G_m} t_m$$

The shear stresses in the fiber, matrix, and composite are assumed to be equal ($\tau_c = \tau_f = \tau_m$), giving

$$\frac{1}{G_{12}} = \frac{1}{G_f}\frac{t_f}{t_c} + \frac{1}{G_m}\frac{t_m}{t_c}$$

Because the thickness fractions are equal to the volume fractions,

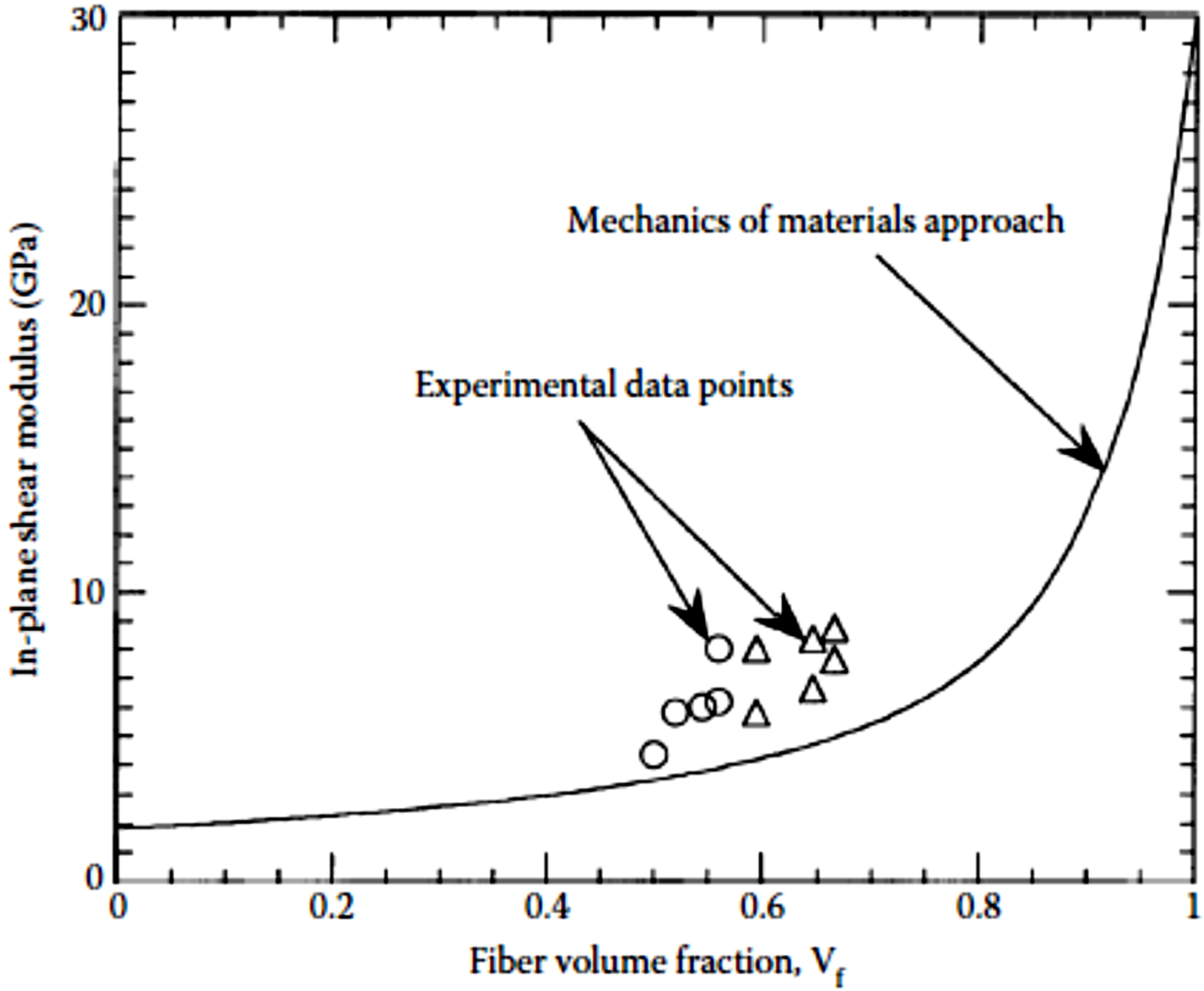
$$\boxed{\frac{1}{G_{12}} = \frac{V_f}{G_f} + \frac{V_m}{G_m}}$$

Or

$$\boxed{\frac{1}{G_{12}} = \frac{V_f}{G_{12f}} + \frac{V_m}{G_m}}$$

For transversely isotropic fiber

In-Plane Shear Modulus, G_{12}



Theoretical values of in-plane shear modulus as a function of fiber volume fraction and comparison with experimental values for a unidirectional glass/epoxy lamina (G_f = 30.19 GPa, G_m = 1.83 GPa). (Experimental data from Hashin, Z., NASA tech. rep. contract No. NAS1-8818.)

3.3. Elastic moduli of a Unidirectional Lamina

Example 3.3

Given a glass/epoxy lamina with a 70% fiber volume fraction. Use properties of glass and epoxy from Tables 3.1 and 3.2 (A.K. Kaw. Mechanics of Composite Materials. 2nd edition, Taylor & Francis, 2006), respectively. Find

1. The longitudinal Young's modulus
2. The transverse Young's modulus
3. The major and minor Poisson's ratios
4. The in-plane shear modulus

Solution

From Table 3.1 the Young's modulus and Poisson's ratios of glass fiber are

$$E_f = 85\,GPa \qquad \nu_f = 0.2 \qquad G_f = 35.42\,GPa$$

From Table 3.2, Young's modulus and Poisson's ratios of epoxy matrix are

$$E_m = 3.4\,GPa \qquad \nu_m = 0.3 \qquad G_m = 1.308\,GPa$$

1. The longitudinal Young's modulus is

$$E_1 = = E_f V_f + E_m V_m = (85)(0.7) + (3.4)(0.3) = 60.52\,GPa$$

3.3. Elastic moduli of a Unidirectional Lamina

2. The transverse Young's modulus is

$$\frac{1}{E_2} = \frac{V_f}{E_f} + \frac{V_m}{E_m} = \frac{0.7}{85} + \frac{0.3}{3.4} \quad \Rightarrow \quad E_2 = 10.37\ GPa$$

3. The major and minor Poisson's ratios are

$$\nu_{12} = = \nu_f V_f + \nu_m V_m = (0.2)(0.7) + (0.3)(0.3) = 0.230$$

$$\nu_{21} = \nu_{12} \frac{E_2}{E_1} = 0.230 \frac{10.37}{60.52} = 0.03941$$

4. The in-plane shear modulus of the unidirectional lamina is

$$\frac{1}{G_{12}} = \frac{V_f}{G_f} + \frac{V_m}{G_m} = \frac{0.70}{35.42} + \frac{0.30}{1.308} \quad \Rightarrow \quad G_{12} = 4.014\ GPa$$

2. Semi-Empirical Models: Halphin-Tsai Equation

The values obtained for transverse Young's modulus and in-plane shear modulus through do not agree well with the experimental results. **Semi-empirical models** have been developed for design purposes. The most useful of these models include those of Halphin and Tsai because they can be used over a wide range of elastic properties and fiber volume fractions.

Longitudinal Young's Modulus

The Halphin–Tsai equation for the Longitudinal Young's Modulus, E_1, is the same as that obtained using the mechanics of materials approach — that is.

$$\boxed{E_1 = E_f V_f + E_m V_m}$$

Major Poisson's Ratio

The Halphin–Tsai equation for the major Poisson's ratio, ν_{12}, is the same as that obtained using the mechanics of materials approach — that is.

$$\boxed{\nu_{12} = \nu_f V_f + \nu_m V_m}$$

2. Semi-Empirical Models: Halphin-Tsai Equation

Transverse Young's Modulus

The transverse Young's modulus, E_2, is given by

$$E_2 = \left(\frac{1 + \xi \eta V_f}{1 - \eta V_f} \right) E_m$$

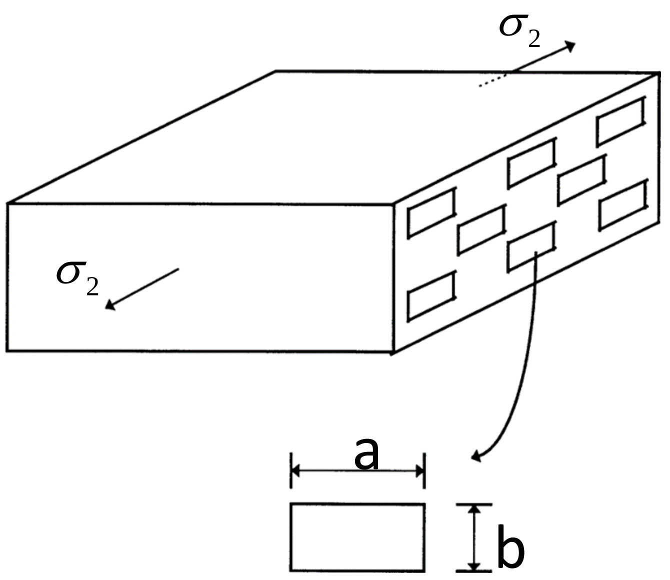
where

$$\eta = \frac{(E_f / E_m) - 1}{(E_f / E_m) + \xi}$$

The term ξ is called the reinforcing factor and depends on the following:

- Fiber geometry
- Packing geometry
- Loading conditions

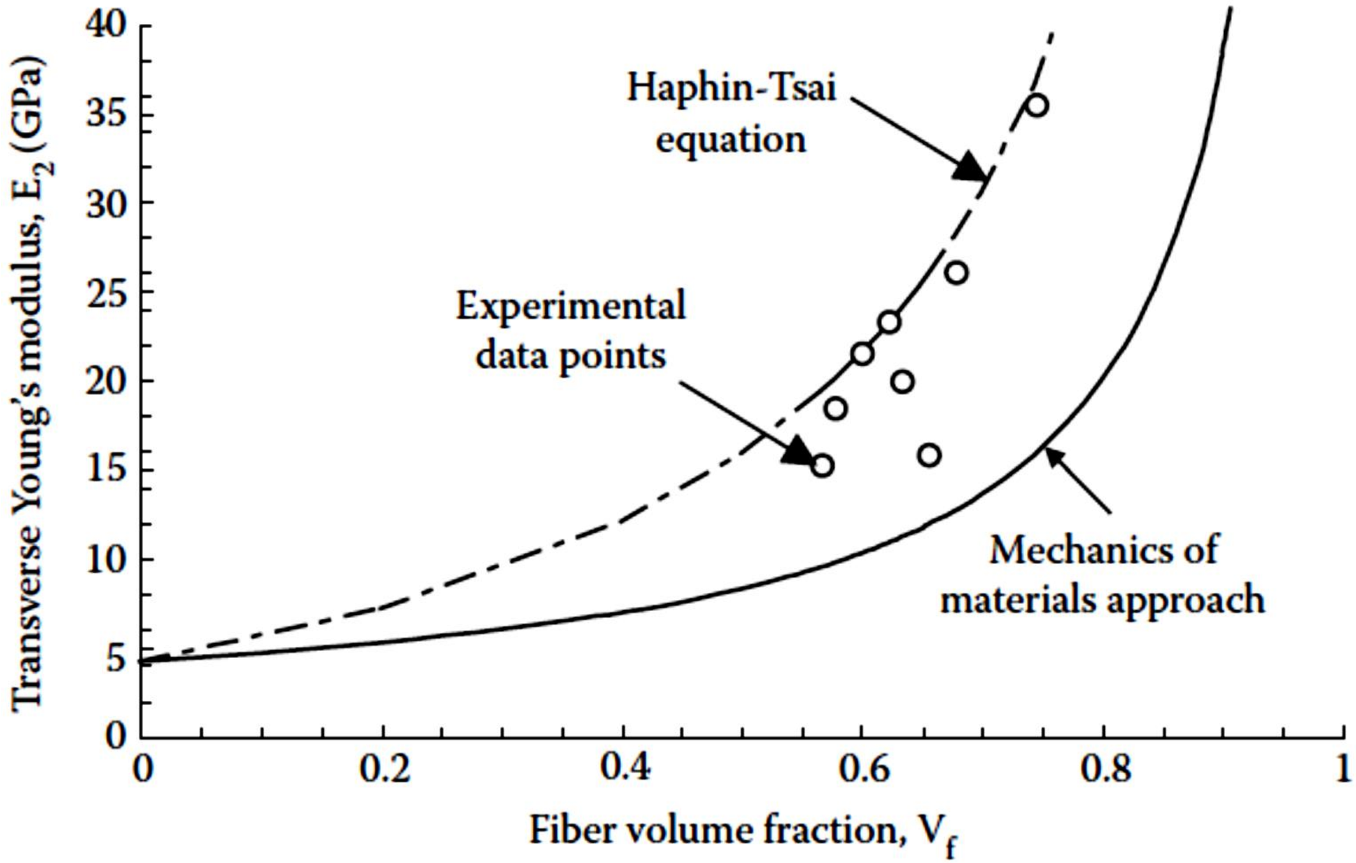
For a fiber geometry of circular fibers in a packing geometry of a square array, $\xi = 2$. For a rectangular fiber cross-section of length a and width b in a hexagonal array, $\xi = 2(a/b)$.



Concept of direction of loading for calculation of transverse Young's modulus by Halphin–Tsai equations.

3.3. Elastic moduli of a Unidirectional Lamina

2. Semi-Empirical Models: Halphin-Tsai Equation



Theoretical values of transverse Young's modulus as a function of fiber volume fraction and comparison with experimental values for boron/epoxy unidirectional lamina (E_f = 414 GPa, ν_f = 0.2, E_m = 4.14 GPa, ν_m = 0.35). Experimental data from Hashin, Z., NASA tech. rep. contract no. NAS1-8818.

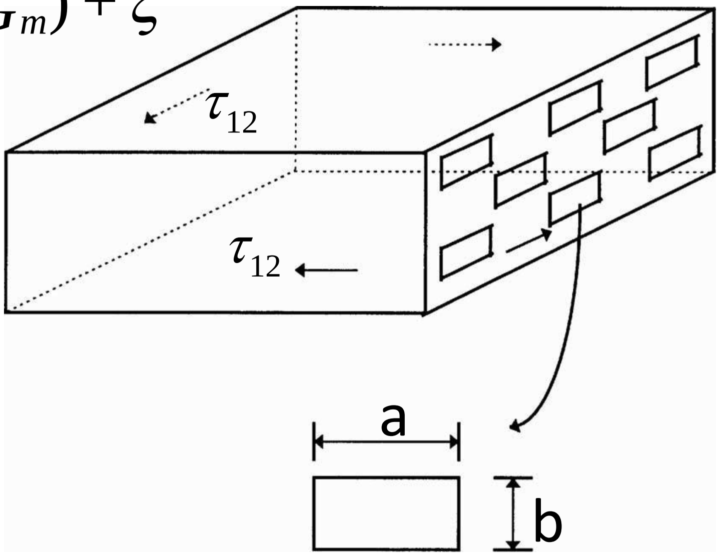
2. Semi-Empirical Models: Halphin-Tsai Equation

In-Plane Shear Modulus

The Halphin–Tsai equation for the in-plane shear modulus, G_{12}, is.

$$G_{12} = \left(\frac{1 + \xi \eta V_f}{1 - \eta V_f} \right) G_m \quad \text{where} \quad \eta = \frac{(G_f / G_m) - 1}{(G_f / G_m) + \xi}$$

For circular fibers in a square array, ξ = 1.
For a rectangular fiber cross-sectional area of length a and width b in a hexagonal array, $\xi = \sqrt{3}\, log_e(a/b)$, where *a* is the direction of loading.

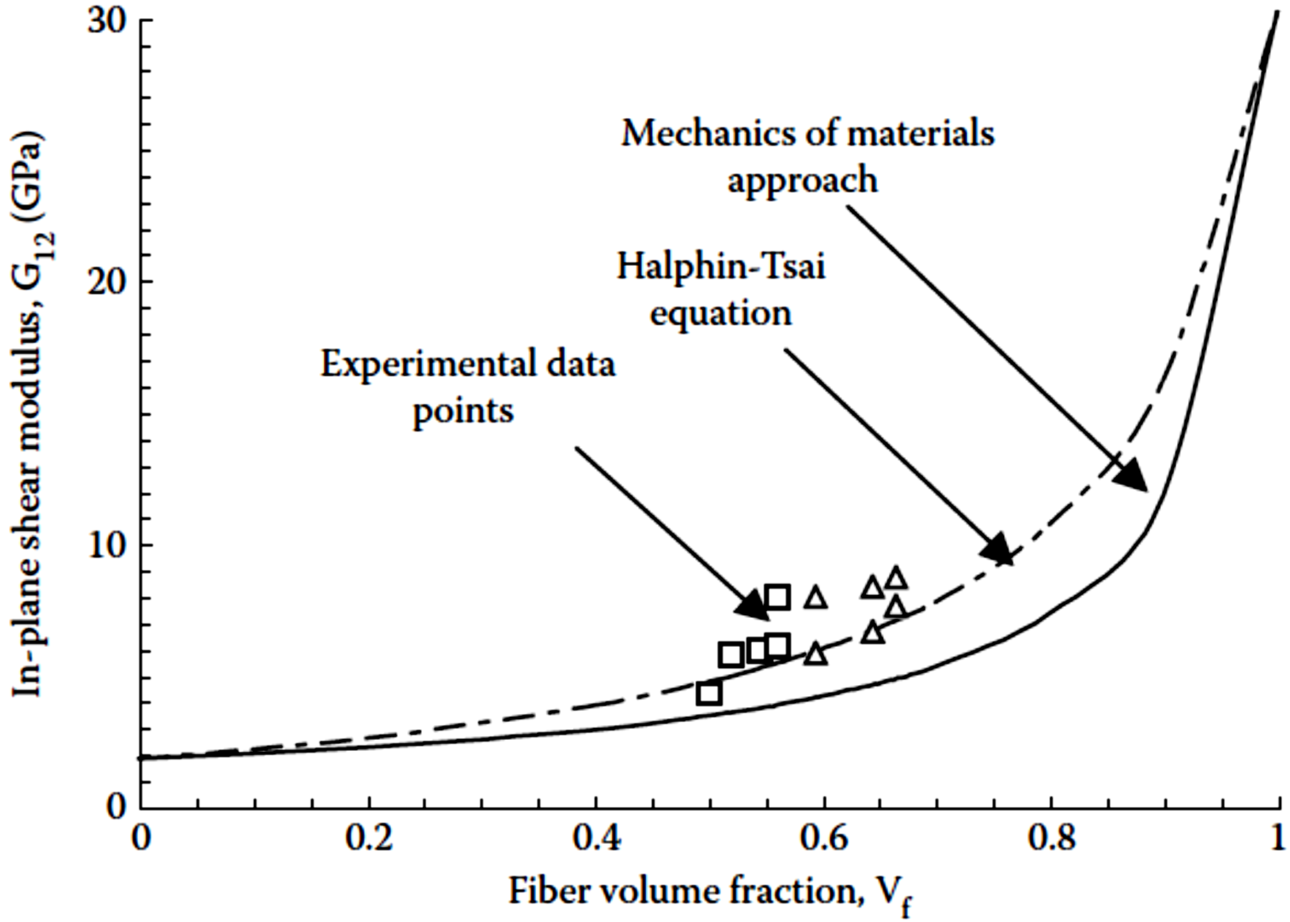


Concept of direction of loading to calculate in-plane shear modulus by Halphin–Tsai equations.

Besides, Hewitt and Malherbe suggested choosing a function,

$$\xi = 1 + 40 V_f^{10}$$

2. Semi-Empirical Models: Halphin-Tsai Equation



Theoretical values of in-plane shear modulus as a function of fiber volume fraction compared with experimental values for unidirectional glass/epoxy lamina (G_f = 30.19 GPa, G_m = 1.83 GPa).

Elastic Moduli of Lamina with Transversely Isotropic Fibers

The elastic moduli using mechanics of materials approach for lamina with transversely isotropic fibers are calculated using following formulas.

$$E_1 = E_{1f} V_f + E_m V_m$$

$$\frac{1}{E_2} = \frac{V_f}{E_{2f}} + \frac{V_m}{E_m}$$

$$\nu_{12} = \nu_{12f} V_f + \nu_m V_m$$

$$\frac{1}{G_{12}} = \frac{V_f}{G_{12f}} + \frac{V_m}{G_m}$$

- E_{1f} = longitudinal (axial) Young's modulus
- E_{2f} = transverse Young's modulus (modulus in plane of isotropy)
- ν_{12f} = axial Poisson's ratio characterizing the contraction in the plane of isotropy when longitudinal tension is applied
- ν_{21f} = transverse Poisson's ratio characterizing the contraction in the longitudinal direction when tension is applied in the plane of isotropy
- G_{12f} = in-plane shear modulus in the plane perpendicular to the plane of isotropy

$$\frac{E_2}{E_m} = \frac{1 + \xi \eta V_f}{1 - \eta V_f}$$

$$\frac{G_{12}}{G_m} = \frac{1 + \xi \eta V_f}{1 - \eta V_f}$$

3.4. Strengths of a Unidirectional Lamina

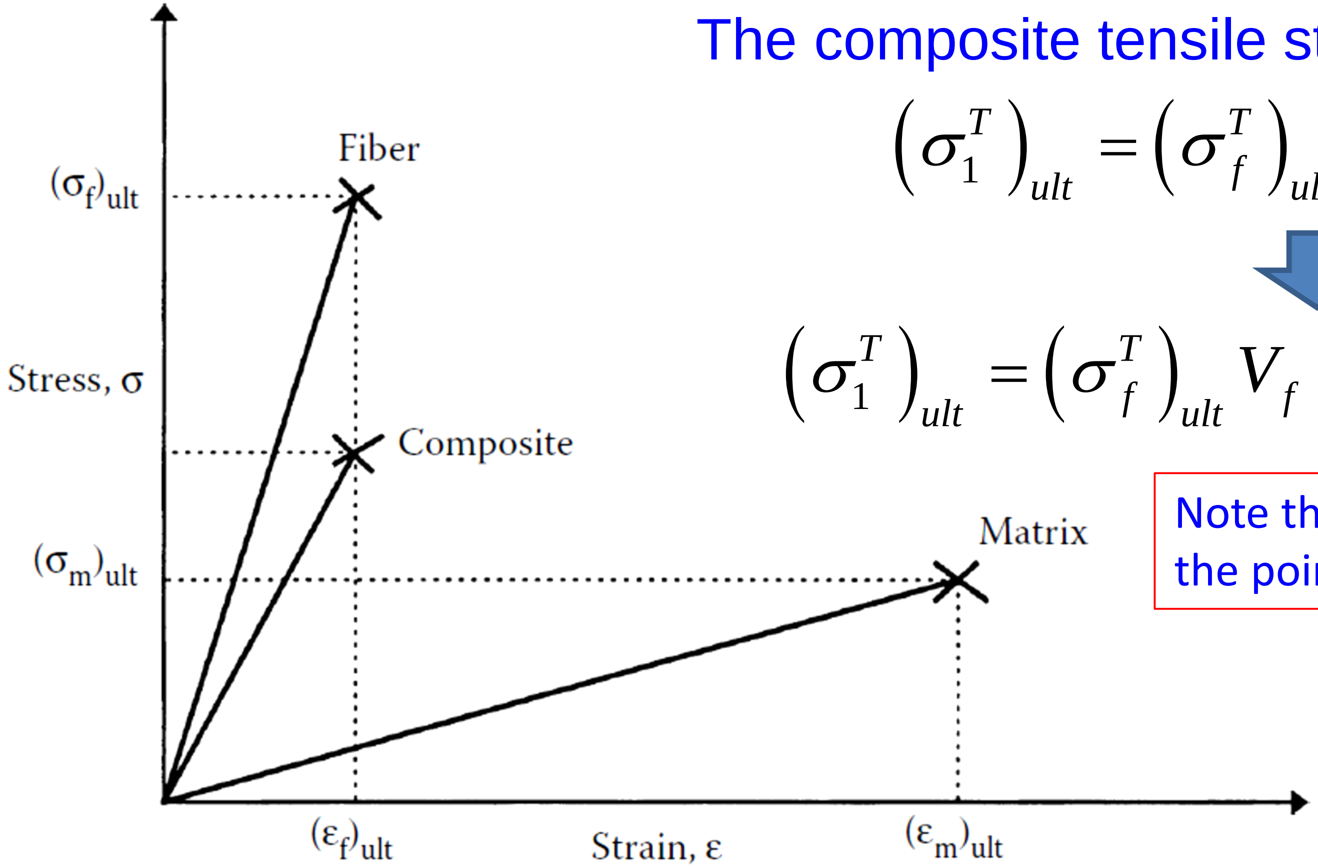
Five ultimate strengths for a unidirectional lamina:

- Longitudinal tensile strength $\left(\sigma_1^T\right)_{ult}$
- Longitudinal compressive strength $\left(\sigma_1^C\right)_{ult}$
- Transverse tensile strength $\left(\sigma_2^T\right)_{ult}$
- Transverse compressive strength $\left(\sigma_2^C\right)_{ult}$
- In-plane shear strength $\left(\tau_{12}\right)_{ult}$

Main assumptions:

- All fibers have the same strength
- Fibers and matrix are isotropic, homogeneous, and linearly elastic until failure.
- Fibers are brittle when compared to matrix
- Fibers are stiffer than matrix

Longitudinal Tensile Strength



Stress–strain curve for a unidirectional composite under uniaxial tensile load along fibers.

The composite tensile strength is given by

$$\left(\sigma_1^T\right)_{ult} = \left(\sigma_f^T\right)_{ult} V_f + \left(\sigma_m^T\right)_{ult} V_m$$

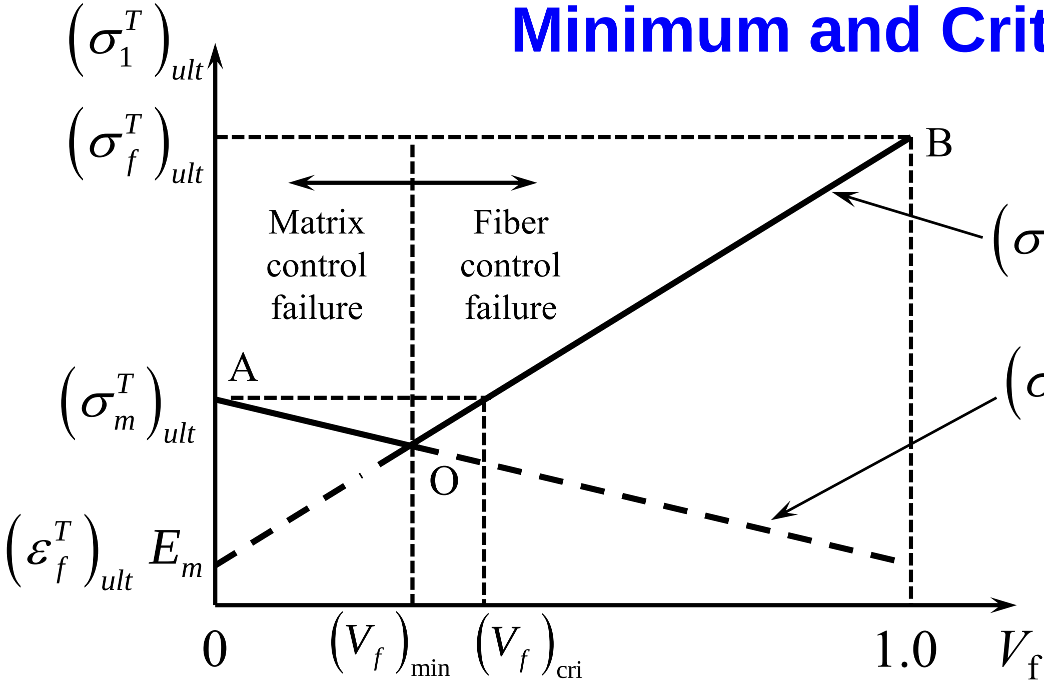
$$\left(\sigma_1^T\right)_{ult} = \left(\sigma_f^T\right)_{ult} V_f + \left(\varepsilon_f^T\right)_{ult} E_m \left(1 - V_f\right)$$

Note that stress in the matrix at the point of maximum fiber strain.

$$\left(\varepsilon_f^T\right)_{ult} = \frac{\left(\sigma_f^T\right)_{ult}}{E_f}$$

$$\left(\varepsilon_m^T\right)_{ult} = \frac{\left(\sigma_m^T\right)_{ult}}{E_m}$$

Minimum and Critical V_f



$$\left(\sigma_1^T\right)_{ult} = \left(\sigma_f^T\right)_{ult} V_f + \left(\varepsilon_f^T\right)_{ult} E_m \left(1 - V_f\right)$$

$$\left(\sigma_1^T\right)_{ult} = \left(\sigma_m^T\right)_{ult} \left(1 - V_f\right)$$ Matrix alone

$$\left(V_f\right)_{\mathrm{minimum}} \rightarrow \left(V_f\right)_{\min}$$

$$\left(V_f\right)_{\mathrm{critical}} \rightarrow \left(V_f\right)_{\mathrm{cri}}$$

$$\left(\sigma_m^T\right)_{ult} [1 - \left(V_f\right)_{\min}] > \left(\sigma_f^T\right)_{ult} \left(V_f\right)_{\min} + \left(\varepsilon_f^T\right)_{ult} E_m \left[1 - \left(V_f\right)_{\min}\right]$$

$$\Rightarrow \boxed{\left(V_f\right)_{min} < \frac{\left(\sigma_m^T\right)_{ult} - E_m \left(\varepsilon_f^T\right)_{ult}}{\left(\sigma_f^T\right)_{ult} - E_m \left(\varepsilon_f^T\right)_{ult} + \left(\sigma_m^T\right)_{ult}}}$$

$$\left(\sigma_m^T\right)_{ult} > \left(\sigma_f^T\right)_{ult} \left(V_f\right)_{\mathrm{cri}} + \left(\varepsilon_f^T\right)_{ult} E_m \left[1 - \left(V_f\right)_{\mathrm{cri}}\right] \Rightarrow \boxed{\left(V_f\right)_{\mathrm{cri}} < \frac{\left(\sigma_m^T\right)_{ult} - E_m \left(\varepsilon_f^T\right)_{ult}}{\left(\sigma_f^T\right)_{ult} - E_m \left(\varepsilon_f^T\right)_{ult}}}$$

Example 3.4

Find the ultimate tensile strength for a Glass/Epoxy lamina with a 70% fiber volume fraction. Use the properties for glass and epoxy from Tables 3.1 and 3.2 (A.K. Kaw. Mechanics of Composite Materials. 2nd edition, Taylor & Francis, 2006), respectively. Also, find the minimum and critical fiber volume fractions.

Solution

From Table 3.1, we have $E_f = 85\,GPa,$ and $(\sigma_f^T)_{ult} = 1550\ MPa$

$$\Rightarrow\ (\varepsilon_f^T)_{ult} = \frac{\left(\sigma_f^T\right)_{ult}}{E_f} = \frac{1550\times 10^6}{85\times 10^9} = 0.1823\times 10^{-1}$$

From Table 3.2, we have $E_m = 3.4\,GPa,$ and $(\sigma_m^T)_{ult} = 72\ MPa$

$$\Rightarrow\ \left(\varepsilon_m^T\right) = \frac{\left(\sigma_m^T\right)_{ult}}{E_m} = \frac{72\times 10^6}{3.4\times 10^9} = 0.2117\times 10^{-1}$$

Ultimate tensile strength for a Glass/Epoxy lamina

$$\left(\sigma_1^T\right)_{ult} = \left(\sigma_f^T\right)_{ult} V_f + \left(\varepsilon_f^T\right)_{ult} E_m \left(1 - V_f\right)$$

$$(\sigma_1^T)_{ult} = (1550\times10^6)(0.7) + (0.1823\times10^{-1})(3.4\times10^9)(1-0.7) = 1104\ MPa$$

Minimum and critical fiber volume fractions

The minimum fiber volume fraction is

$$(V_f)_{min} = \frac{72\times10^6 - (3.4\times10^9)(0.1823\times10^{-1})}{1550\times10^6 - (3.4\times10^9)(0.1823\times10^{-1}) + 72\times10^6}$$

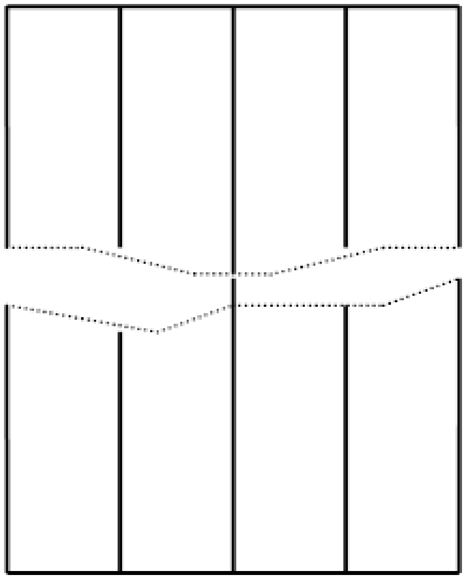
$$= 0.6422\times10^{-2} = 0.6422\%$$

The critical fiber volume fraction is

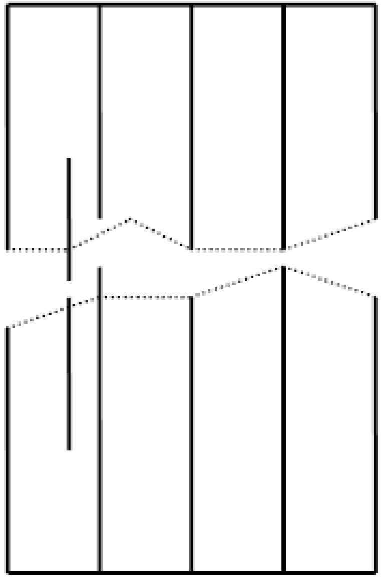
$$(V_f)_{cri} = \frac{72\times10^6 - (3.4\times10^9)(0.1823\times10^{-1})}{1550\times10^6 - (3.4\times10^9)(0.1823\times10^{-1})}$$

$$= 0.6732\times10^{-2} = 0.6732\%$$

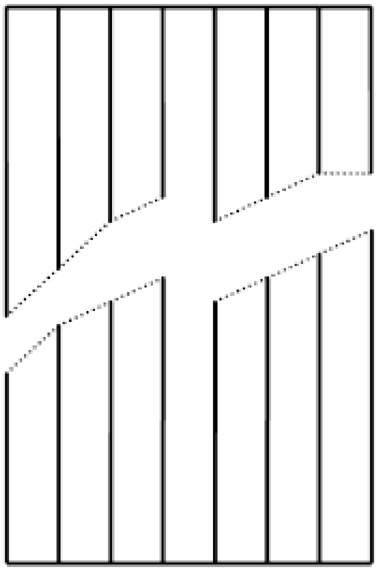
Modes of failure of unidirectional lamina under a longitudinal tensile load



(a)



(b)

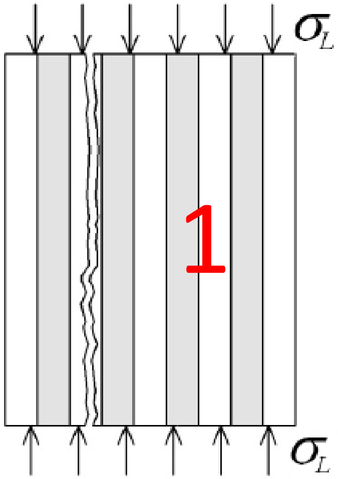


(c)

a) Brittle fracture of fibers ($0 < V_f < 0.40$)
b) Brittle fracture of fibers with pullout ($0.4 < V_f < 0.65$)
c) Fiber pullout with fiber–matrix debonding ($V_f > 0.65$)

Longitudinal Compressive Strength

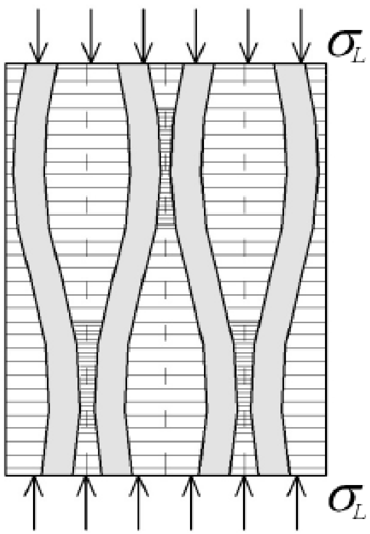
Three typical failure modes of a unidirectional lamina under a longitudinal compressive load



1. Fracture of matrix and/or fiber–matrix bond due to transverse tensile strains in the matrix and/or bond

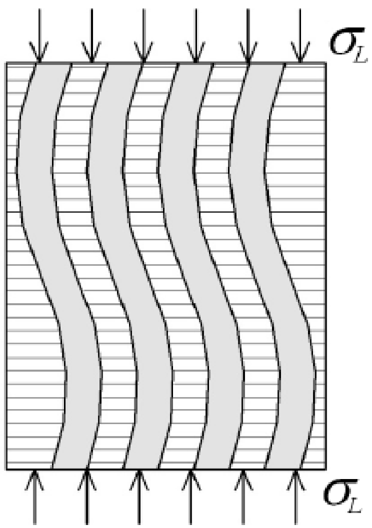
2. Micro-buckling of fibers in shear or extensional mode

3. Shear failure of fibers

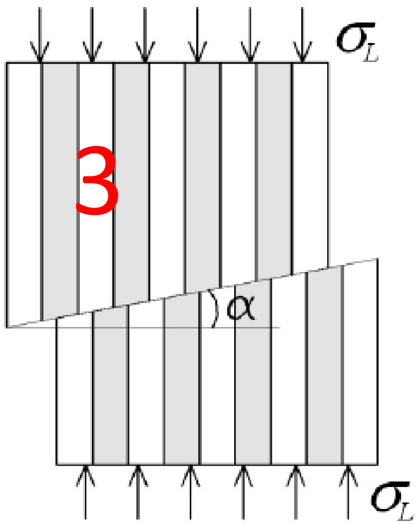


Extensional mode

2



Shear mode



1. Fracture of matrix due to tensile strains in the matrix

Assuming that one is applying a longitudinal compressive stress of magnitude σ_1, then the magnitude of longitudinal compressive strain is given by

$$|\varepsilon_1| = \frac{|\sigma_1|}{E_1}$$

Because the major Poisson's ratio is ν_{12}, the transverse strain is tensile and is given by

$$\nu_{12} = \frac{|\varepsilon_2|}{|\varepsilon_1|} \Rightarrow |\varepsilon_2| = \nu_{12}\frac{|\sigma_1|}{E_1}$$

Using maximum strain failure theory, if the transverse strain exceeds the ultimate transverse tensile strain, $(\varepsilon_2^T)_{ult}$ the lamina is considered to have failed in the transverse direction. Thus,

$$\boxed{(\sigma_1^c)_{ult} = \frac{E_1(\varepsilon_2^T)_{ult}}{\nu_{12}}}$$

For the value $(\varepsilon_2^T)_{ult}$ of one can use the empirical formula or the mechanics of materials formula,

$$(\varepsilon_2^T)_{ult} = (\varepsilon_m^T)_{ult}(1 - V_f^{1/3})$$

$$(\varepsilon_2^T)_{ult} = (\varepsilon_m^T)_{ult}\left[\frac{d}{s}\left(\frac{E_m}{E_f} - 1\right) + 1\right]$$

d = diameter of the fibers

s = center-to-center spacing between the fibers

For circular fibers in a square array, $\frac{d}{s} = \left[\frac{4V_f}{\pi}\right]^{\frac{1}{2}}$

For circular fibers in a hexagonal array, $\frac{d}{s} = \left[\frac{2\sqrt{3}V_f}{\pi}\right]^{\frac{1}{2}}$

2. Micro-buckling of fibers in shear or extensional mode

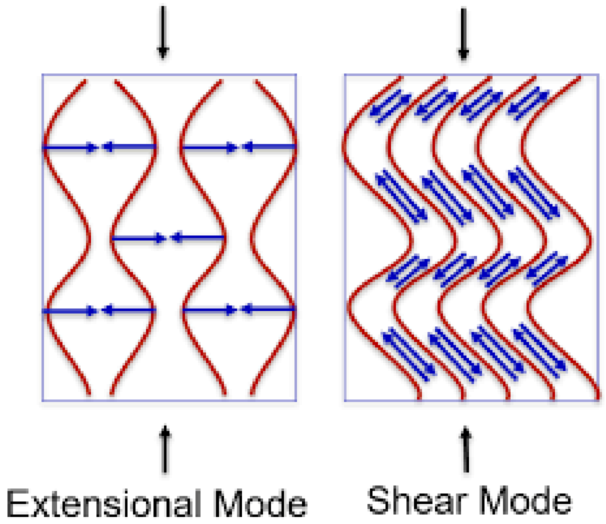
Ultimate longitudinal compressive strength

$$\boxed{(\sigma_1^c)_{ult} = \min[S_1^c, S_2^c]}$$

where

$$S_1^c = 2\left[V_f + (1-V_f)\frac{E_m}{E_f}\right]\sqrt{\frac{V_f E_m E_f}{3(1-V_f)}}, \text{and}$$

$$S_2^c = \frac{G_m}{1-V_f}$$



Note that the extensional mode buckling stress (S_1^c) is higher than the shear mode buckling stress (S_2^c) for most cases. Extensional mode buckling is prevalent only in low fiber volume fraction composites.

3. Shear stress failure of fibers mode

A unidirectional composite may fail due to direct shear failure of fibers. In this case, the rule of mixtures gives the shear strength of the unidirectional composite as

$$(\tau_{12})_{ult} = (\tau_f)_{ult} V_f + (\tau_m)_{ult} V_m$$

where

$(\tau_f)_{ult}$ = ultimate shear strength of the fiber

$(\tau_m)_{ult}$ = ultimate shear strength of the matrix

The maximum shear stress in a lamina under a longitudinal compressive load $\left(\sigma_1^c\right)$ is $(\tau_{12})_{ult} = \left(\sigma_1^c\right)/2$ at 45° to the loading axis. Thus,

$$\boxed{(\sigma_1^c)_{ult} = 2[(\tau_f)_{ult} V_f + (\tau_m)_{ult} V_m]}$$

Example 3.5

Find the longitudinal compressive strength of a Glass/Epoxy lamina with a 70% fiber volume fraction. Use the properties of glass and epoxy from Tables 3.1 and 3.2 (A.K. Kaw. Mechanics of Composite Materials. 2nd edition, Taylor & Francis, 2006). Assume fibers are circular and are in a square array.

Solution

From Table 3.1

$$E_f = 85\ GPa,$$

$$\nu_f = 0.20,$$

$$(\sigma_f)_{ult} = 1550\ MPa,$$

and

$$(\tau_f)_{ult} = 35\ MPa$$

From Table 3.2

$$E_m = 3.4\ GPa,$$

$$\nu_m = 0.30,$$

$$(\sigma_m)_{ult} = 72\ MPa,$$

and

$$(\tau_m)_{ult} = 34\ MPa$$

The longitudinal elastic modulus $E_1 = E_f V_f + E_m V_m = 60.50\ GPa$

The major Poisson's ratio $\nu_{12} = \nu_f V_f + \nu_m V_m = 0.23$

The fiber diameter to fiber spacing ratio is $\dfrac{d}{s} = \left[\dfrac{4(0.7)}{\pi}\right]^{\frac{1}{2}} = 0.9441$

1. Using fracture of matrix due to tensile strains in the matrix

The ultimate tensile strain of the matrix is

$$\left(\varepsilon_m^T\right)_{ult} = \frac{\left(\sigma_m^T\right)_{ult}}{E_m} = \frac{72\times10^6}{3.40\times10^9} = 0.2117\times10^{-1}$$

The transverse ultimate tensile strain failure mode

$$(\varepsilon_2^T)_{ult} = 0.2117\times10^{-1}\left[0.9441\left(\frac{3.4\times10^9}{85\times10^9} - 1\right) + 1\right] = 0.1983\times10^{-2}$$

From the empirical equation, transverse ultimate tensile strain is

$$(\varepsilon_2^T)_{ult} = (0.2117\times10^{-1})\,(1 - 0.7^{1/3}) = 0.2373\times10^{-2}$$

3.4. Strengths of a Unidirectional Lamina

Using the lesser of these two values of ultimate transverse tensile strain, ultimate compressive strength is

$$(\sigma_1^C)_{ult} = \frac{(60.52\times 10^9)\,(0.1983\times 10^{-2})}{0.23} = 521.8\ MPa$$

2. Using shear/extensional fiber microbuckling failure mode

$$S_1^C = 2\left[0.7 + (1-0.7)\,\frac{3.4\times 10^9}{85\times 10^9}\right]\sqrt{\frac{(0.7)(3.4\times 10^9)(85\times 10^9)}{3(1-0.7)}} = 21349\ MPa$$

$$G_m = 1.308\,GPa \quad \Rightarrow \quad S_2^C = \frac{1.308\times 10^9}{1-0.7} = 4360\ MPa$$

Ultimate compressive strength is

$$(\sigma_1^C)_{ult} = \min(21349, 4360) = 4360\ MPa$$

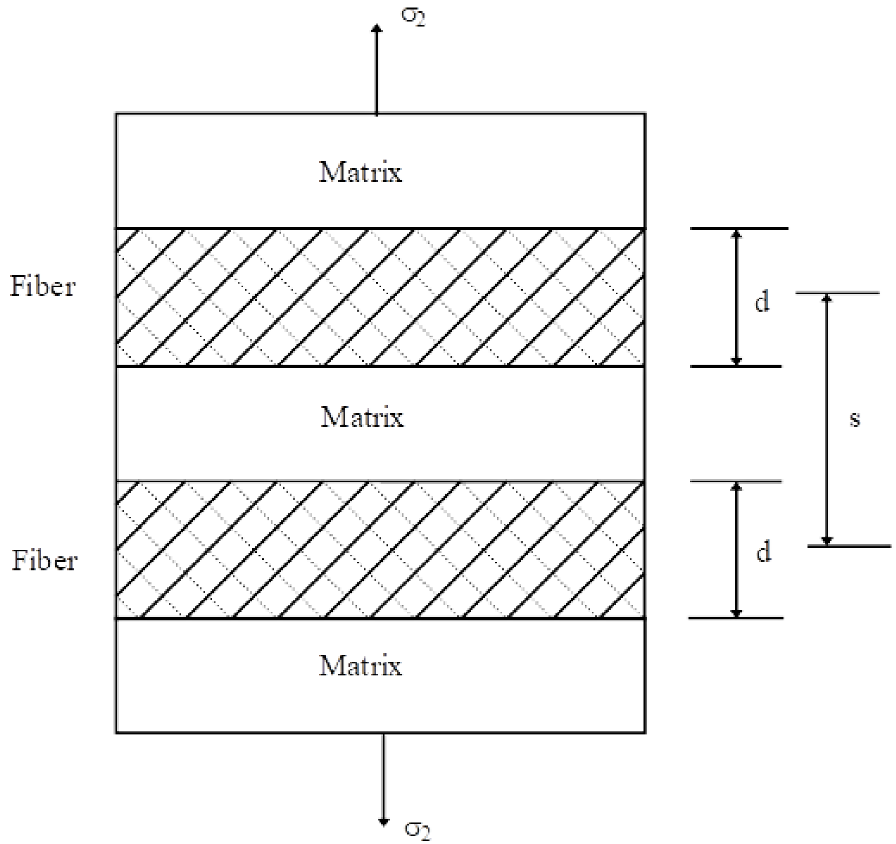
3. Using shear stress failure of fibers mode

$$(\sigma_1^C)_{ult} = 2[(35\times 10^6)(0.7) + (34\times 10^6)(0.3)] = 69.4\ MPa$$

Taking the minimum value

$$(\sigma_1^C)_{ult} = 69.4\ MPa$$

Transverse Tensile Strength



Representative volume element to calculate transverse tensile strength of a unidirectional lamina.

Assumptions used in the mechanics of materials approach model include

- A perfect fiber–matrix bond
- Uniform spacing of fibers
- The fiber and matrix follow Hooke's law
- There are no residual stresses

The ultimate transverse tensile strength is given by

$$\boxed{(\sigma_2^T)_{ult} = E_2 (\varepsilon_2^T)_{ult}}$$

For the value $(\varepsilon_2^T)_{ult}$ of one can use the empirical formula or the mechanics of materials formula.

The ultimate transverse tensile strain

The transverse deformations of the fiber, δ_f, the matrix, δ_m, and the composite, δ_c, are related by

$$\delta_c = \delta_f + \delta_m$$

where $\delta_c = s\varepsilon_c, \quad \delta_f = d\varepsilon_f$, and $\delta_m = (s-d)\,\varepsilon_m$

$$\Rightarrow \quad \varepsilon_c = \frac{d}{s}\varepsilon_f + \left(1 - \frac{d}{s}\right)\varepsilon_m$$

$$\underbrace{E_f\varepsilon_f}_{\sigma_f} = \underbrace{E_m\varepsilon_m}_{\sigma_m} \longrightarrow \varepsilon_c = \left[\frac{d}{s}\frac{E_m}{E_f} + \left(1 - \frac{d}{s}\right)\right]\varepsilon_m$$

The stresses in the fiber and matrix are equal.

$$\boxed{(\varepsilon_2^T)_{ult} = \left[\frac{d}{s}\frac{E_m}{E_f} + \left(1 - \frac{d}{s}\right)\right](\varepsilon_m^T)_{ult}}$$

Transverse Compressive Strength

Equation, which was developed for evaluating transverse tensile strength, can be used to find the transverse compressive strengths of a lamina. The actual compressive strength is again lower due to imperfect fiber/matrix interfacial bond and longitudinal fiber splitting. Using compressive parameters, the ultimate transverse compressive strength is

$$\boxed{(\sigma_2^C)_{ult} = E_2 (\varepsilon_2^C)_{ult}}$$

where

$$(\varepsilon_2^C)_{ult} = \left[\frac{d}{s}\frac{E_m}{E_f} + \left(1 - \frac{d}{s}\right)\right](\varepsilon_m^C)_{ult}$$

Example 3.6

Find the ultimate transverse tensile strength and ultimate transverse compressive strength of a glass/epoxy lamina with 70% fiber volume fraction. Use the properties of glass and epoxy from Table 3.1 and Table 3.2 (A.K. Kaw. Mechanics of Composite Materials. 2nd edition, Taylor & Francis, 2006), respectively. Assume the fibers are circular and are packed in a square array.

Solution

1. Ultimate transverse tensile strength

From Example 3.5, the ultimate transverse tensile strain of the lamina is

$$(\varepsilon_2^T)_{ult} = 0.1983 \times 10^{-2}$$

From Example 3.3 $\quad E_2 = 10.37\ GPa,$

The ultimate transverse tensile strength of the lamina is

$$(\sigma_2^T)_{ult} = E_2 (\varepsilon_2^T)_{ult} = (10.37 \times 10^9)(0.1983 \times 10^{-2}) = 20.56\ \text{MPa}$$

2. Ultimate transverse compressive strength

From Table 3.1 and 3.2 $E_f = 85\ GPa, \quad E_m = 3.4\ GPa,$

$$(\sigma_m^C)_{ult} = 102\ MPa,$$

From Example 3.3 $E_2 = 10.37\ GPa,$

From Example 3.5 $\frac{d}{s} = 0.9441$

The ultimate compressive strain of the matrix is

$$(\varepsilon_m^C)_{ult} = \frac{102 \times 10^6}{3.4 \times 10^9} = 0.0300$$

The ultimate transverse compressive strain of the lamina is

$$(\varepsilon_2^C)_{ult} = \left[0.9441 \frac{3.4 \times 10^9}{85 \times 10^9} + (1 - 0.9441)\right](0.03) = 0.2810 \times 10^{-2}$$

The ultimate transverse compressive strength is

$$(\sigma_2^C)_{ult} = (10.37 \times 10^9)(0.2810 \times 10^{-2}) = 29.14\ MPa$$

In-Plane Shear Strength

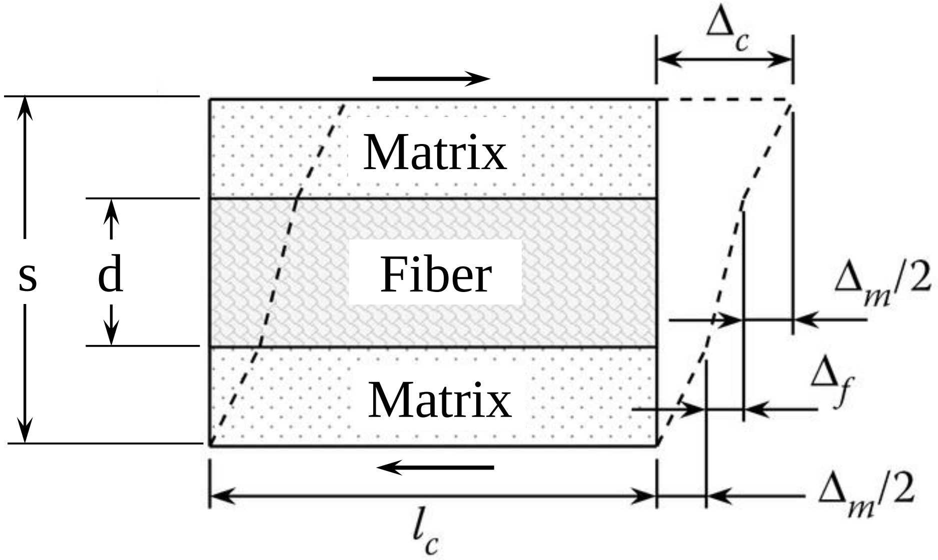
The procedure for finding the in-plane ultimate shear strength for a UD lamina. Assume that one is applying a shear stress of magnitude τ_{12} and then that the shearing deformation in the representative element is given by the sum of the deformations in the fiber and matrix.

$$\Delta_c = \Delta_f + \Delta_m$$

$$\Delta_c = s(\gamma_{12})_c ,$$

$$\Delta_f = d(\gamma_{12})_f ,$$

$$\Delta_m = (s - d)(\gamma_{12})_m$$



$$\Rightarrow \ (\gamma_{12})_c = \frac{d}{s}(\gamma_{12})_f + \left(1 - \frac{d}{s}\right)(\gamma_{12})_m$$

3.4. Strengths of a Unidirectional Lamina

Under shearing stress loading, one assumes that the shear stress in the fiber and matrix are equal. Then, the shear strains in the fiber and matrix are related as

$$(\gamma_{12})_m G_m = (\gamma_{12})_f G_f$$

$$\Rightarrow \quad (\gamma_{12})_c = \left[\frac{d}{s}\frac{G_m}{G_f} + \left(1 - \frac{d}{s}\right)\right](\gamma_{12})_m$$

If one assumes that the shear failure is due to failure of the matrix, then

$$(\gamma_{12})_{ult} = \left[\frac{d}{s}\frac{G_m}{G_f} + \left(1 - \frac{d}{s}\right)\right](\gamma_{12})_{m\,ult}$$

The ultimate shear strength is then given by

$$\boxed{(\tau_{12})_{ult} = G_{12}(\gamma_{12})_{ult} = G_{12}\left[\frac{d}{s}\frac{G_m}{G_f} + \left(1 - \frac{d}{s}\right)\right](\gamma_{12})_{m\ ult}}$$

Example 3.7: Find the ultimate shear strength for a Glass/Epoxy lamina with 70% fiber volume fraction. Use properties for glass and epoxy from Tables 3.1 and 3.2 (A.K. Kaw. Mechanics of Composite Materials. 2nd edition, Taylor & Francis, 2006), respectively. Assume the fibers are circular and are arranged in a square array.

Solution

From Example 3.3: $G_f = 35.42\,GPa \quad G_m = 1.308\,GPa \quad G_{12} = 4.014\,GPa$

From Example 3.5: $\dfrac{d}{s} = 0.9441$

From Table 3.2: $(\tau_{12})_{m\,ult} = 34\ MPa$

The ultimate shear strain of the matrix is $(\gamma_{12})_{m\,ult} = \dfrac{34 \times 10^6}{1.308 \times 10^9} = 0.2599 \times 10^{-1}$

The ultimate in-plane shear strength of the unidirectional lamina is

$$(\tau_{12})_{ult} = G_{12}(\gamma_{12})_{ult} = G_{12}\left[\frac{d}{s}\frac{G_m}{G_f} + \left(1 - \frac{d}{s}\right)\right](\gamma_{12})_{m\ ult} = 9.469\,MPa$$

Strengths of Lamina with Transversely Isotropic Fibers

The strength using mechanics of materials approach for lamina with transversely isotropic fibers are calculated using following formulas.

$$(\sigma_1^c)_{ult} = \frac{E_1 (\varepsilon_2^T)_{ult}}{\nu_{12}}$$

$$(\varepsilon_2^T)_{ult} = (\varepsilon_m^T)_{ult} \left[\frac{d}{s} \left(\frac{E_m}{E_{2f}} - 1 \right) + 1 \right]$$

$$S_1^c = 2\left[V_f + (1 - V_f) \frac{E_m}{E_{1f}}\right] \sqrt{\frac{V_f E_m E_{1f}}{3(1 - V_f)}}$$

$$(\sigma_2^T)_{ult} = E_2 (\varepsilon_2^T)_{ult}$$

$$(\varepsilon_2^T)_{ult} = \left[\frac{d}{s} \frac{E_m}{E_{2f}} + \left(1 - \frac{d}{s} \right) \right] (\varepsilon_m^T)_{ult}$$

$$(\sigma_2^C)_{ult} = E_2 (\varepsilon_2^C)_{ult}$$

$$(\varepsilon_2^C)_{ult} = \left[\frac{d}{s} \frac{E_m}{E_{2f}} + \left(1 - \frac{d}{s} \right) \right] (\varepsilon_m^C)_{ult}$$

3.5. Coefficients of Thermal Expansion

The linear coefficient of thermal expansion (CTE) is a measure of relative change in dimensions with respect to the change in temperature. For an isotropic material, it is defined as

$$\alpha = \frac{\Delta l}{l \Delta T}$$

Where, α is the CTE of the material, Δl is the change in length, l is the original length, ΔT is the change in temperature

For a unidirectional lamina, the dimensions changes differ in the two directions 1 and 2. Thus, the two linear CTEs are defined as α_1 and α_2 (Unit: m/m/°C or in./in./°F).

- α_1 = linear coefficient of thermal expansion in direction 1 or longitudinal thermal expansion coefficient, m/m/°C (in./ in./°F)
- α_2 = linear coefficient of thermal expansion in direction 2 or transverse thermal expansion coefficient, m/m/°C (in./ in./°F)

Longitudinal Thermal Expansion Coefficient

Consider the expansion of a unidirectional lamina in the longitudinal direction under a temperature change of ΔT. If only the temperature ΔT is applied, the unidirectional lamina has zero overall load, F_1, in the longitudinal direction. Then

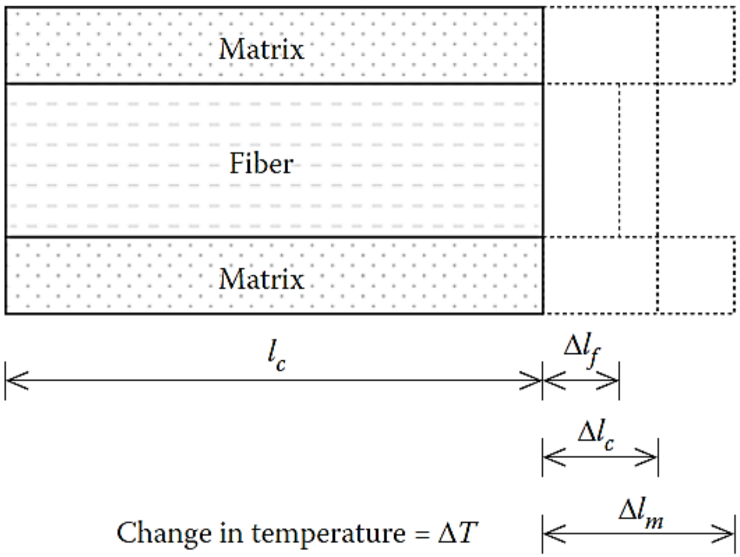
$$F_1 = \sigma_1 A_c = 0 = \sigma_f A_f + \sigma_m A_m$$

$$\Rightarrow \sigma_f V_f + \sigma_m V_m = 0$$

Although the overall load in the longitudinal direction 1 is zero, stresses are caused in the fiber and the matrix by the thermal expansion mismatch between the fiber and the matrix. These stresses are

$$\sigma_f = E_f(\varepsilon_f - \alpha_f \Delta T), \text{ and}$$

$$\sigma_m = E_m(\varepsilon_m - \alpha_m \Delta T)$$



Representative volume element for longitudinal coefficient of thermal expansion- schematic representation of longitudinal deformations due to change in temperature.

Longitudinal Thermal Expansion Coefficient

Realizing that the strains in the fiber and matrix are equal ($\varepsilon_f = \varepsilon_m = \varepsilon_1$),

$$\varepsilon_f = \frac{\alpha_f E_f V_f + \alpha_m E_m V_m}{E_f V_f + E_m V_m} \Delta T$$

For free expansion in the composite in the longitudinal direction 1, the longitudinal strain is $$\varepsilon_1 = \alpha_1 \Delta T$$

Because the strains in the fiber and composite are also equal ($\varepsilon_1 = \varepsilon_f$),

$$\alpha_1 = \frac{\alpha_f E_f V_f + \alpha_m E_m V_m}{E_f V_f + E_m V_m}$$

$$\alpha_1 = \frac{1}{E_1}(\alpha_f E_f V_f + \alpha_m E_m V_m) \Rightarrow \alpha_1 = \left(\frac{\alpha_f E_f}{E_1}\right) V_f + \left(\frac{\alpha_m E_m}{E_1}\right) V_m$$

3.5. Coefficients of Thermal Expansion

Transverse Thermal Expansion Coefficient

Due to temperature change, ΔT, assume that the compatibility condition that the strain in the fiber and matrix is equal in direction 1 that is,

$$\in_m = \in_f = \in_1$$

Now, the stress in the fiber in the longitudinal direction 1 is

$$(\sigma_f)_1 = E_f (\in_f)_1 = E_f \in_1 = E_f (\alpha_1 - \alpha_f)\Delta T$$

and the stress in the matrix in longitudinal direction 1 is

$$(\sigma_m)_1 = E_m (\in_m)_1 = E_m \in_1 = E_m (\alpha_m - \alpha_1)\Delta T$$

The strains in the fiber and matrix in the transverse direction 2 are given by using Hooke's law:

$$(\in_f)_2 = \alpha_f \Delta T - \frac{\nu_f (\sigma_f)_1}{E_f} \qquad (\in_m)_2 = \alpha_m \Delta T - \frac{\nu_m (\sigma_m)_1}{E_m}$$

Transverse Thermal Expansion Coefficient

The transverse strain of the composite is given by the rule of mixtures as,

$$\in_2 = \left(\in_f\right)_2 V_f + \left(\in_m\right)_2 V_m$$

$$\in_2 = \left[\alpha_f \Delta T - \frac{\nu_f E_f\left(\alpha_1 - \alpha_f\right)\Delta T}{E_f}\right] V_f + \left[\alpha_m \Delta T - \frac{\nu_m E_m\left(\alpha_m - \alpha_1\right)\Delta T}{E_m}\right] V_m$$

because $\in_2 = \alpha_2 \Delta T$

$$\Rightarrow \alpha_2 = [\alpha_f - \nu_f(\alpha_1 - \alpha_f)]V_f + [\alpha_m - \nu_m(\alpha_m - \alpha_1)]V_m$$

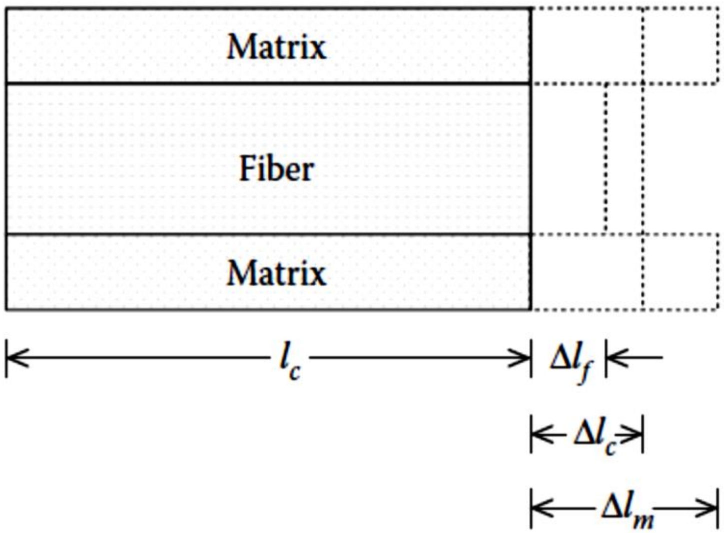
Substituting $\nu_{12} = \nu_f V_f + \nu_m V_m$

$$\boxed{\alpha_2 = \left(1 + \nu_f\right)\alpha_f V_f + \left(1 + \nu_m\right)\alpha_m V_m - \alpha_1 \nu_{12}}$$

3.6. Coefficients of Moisture Expansion

The change in dimensions of the body are measured by the coefficient of moisture expansion (CME) defined as the change in the linear dimension of a body per unit length per unit change in weight of moisture content per unit weight of the body.

$$\beta = \frac{\Delta l}{l \Delta C}$$



Change in moisture content per unit mass of composite = ΔC_c

Where, β is the CME of the material (m/m/kg/kg), Δl is the change in length (m), l is the original length (m), ΔC is the change in in moisture content per unit mass of the body (kg/kg)

Similar to the coefficients of thermal expansion, there are two linear coefficients of moisture expansion: one in the longitudinal direction 1 and the other in the transverse direction 2: β_1 and β_2.

Longitudinal coefficient of moisture expansion

Consider the expansion of a unidirectional lamina in the longitudinal direction because of change in moisture content in the composite. The overall load in the composite, F_1, is zero, that is

$$F_1 = \sigma_1 A_c = 0 = \sigma_f A_f + \sigma_m A_m, \text{ and } \quad \sigma_f V_f + \sigma_m V_m = 0$$

The stresses in the fiber and matrix caused by moisture are

$$\sigma_f = E_f (\varepsilon_f - \beta_f \Delta C_f), \text{and} \quad \sigma_m = E_m (\varepsilon_m - \beta_m \Delta C_m)$$

The strains in the fiber and matrix are equal ($\varepsilon_f = \varepsilon_m$),

$$\varepsilon_f = \frac{\beta_f \Delta C_f V_f E_f + \beta_m \Delta C_m V_m E_m}{E_f V_f + E_m V_m}$$

For free expansion of the composite in the longitudinal direction, the longitudinal strain is

$$\varepsilon_1 = \beta_1 \Delta C_c$$

where ΔC_c = the moisture concentration in composite.

3.6. Coefficients of Moisture Expansion

Because the strains in the fiber and the matrix are equal,

$$\beta_1 = \frac{\beta_f \ \Delta C_f V_f E_f + \beta_m \Delta C_m V_m E_m}{(E_f V_f + E_m V_m)\,\Delta C_c}$$

The moisture content in the composite is the sum of the moisture contents in the fiber and the matrix,

$$\Delta C_c w_c = \Delta C_f w_f + \Delta C_m w_m$$

where $w_{c,f,m}$ = the mass of composite, fiber, and matrix, respectively. Thus,

$$\Delta C_c = \Delta C_f W_f + \Delta C_m W_m$$

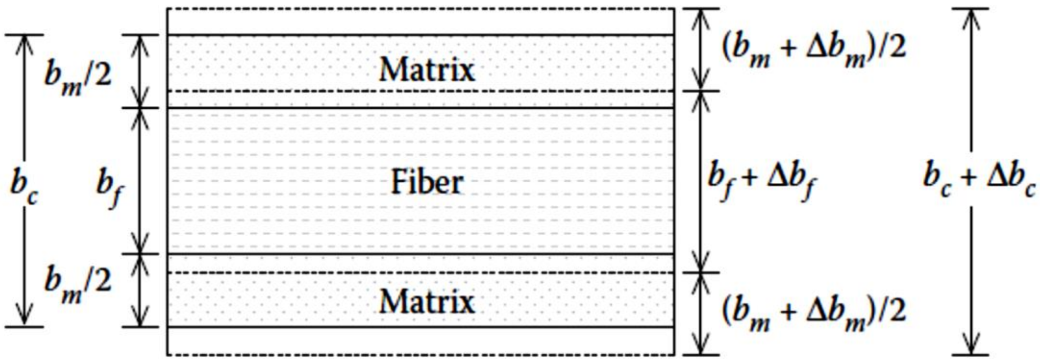
where $W_{f,m}$ = the mass fractions of the fiber and matrix, respectively.

$$\beta_1 = \frac{\beta_f \ \Delta C_f V_f E_f + \beta_m \Delta C_m V_m E_m}{(E_f V_f + E_m V_m)(\Delta C_f W_f + \Delta C_m W_m)}$$

$$\boxed{or \quad \beta_1 = \frac{\beta_f \Delta C_f V_f E_f + \beta_m \Delta C_m V_m E_m}{E_1(\Delta C_f \rho_f V_f + \Delta C_m \rho_m V_m)} \rho_c}$$

Transverse coefficient of moisture expansion

In the case of transverse CME, the methodology is similar to that for the transverse CTE and we equate the total transverse expansion to the sum of the expansions of the fibers and matrix, that is, $\Delta b_c = \Delta b_f + \Delta b_m$ ⇒ $\beta_2 \Delta C_c b_c = \beta_f \Delta C_f b_f + \beta_m \Delta C_m b_m$



Change in moisture content per unit mass of composite = ΔC_c

Dividing both the sides by $\Delta C_c b_c$, we get $\beta_2 = \dfrac{\beta_f \Delta C_f V_f + \beta_m \Delta C_m V_m}{\Delta C_c}$

Substituting ΔC_c, the transverse CMEs will be

$$\beta_2 = \left[\frac{\beta_f \Delta C_f V_f + \beta_m \Delta C_m (1 - V_f)}{\Delta C_f \rho_f V_f + \Delta C_m \rho_m (1 - V_f)}\right]\left[\rho_f V_f + \rho_m (1 - V_f)\right]$$

or $$\beta_2 = \frac{V_f (1 + \nu_f) \Delta C_f \beta_f + V_m (1 + \nu_m) \Delta C_m \beta_m}{(V_m \rho_m \Delta C_m + V_f \rho_f \Delta C_f)} \rho_c - \beta_1 \nu_{12}$$

3.6. Coefficients of Moisture Expansion

In most polymeric matrix composites, fibers do not absorb or absorb a little moisture, so the expressions for coefficients of moisture expansion do become independent of moisture contents. Substituting $\Delta C_f = 0$

$$\beta_1 = \frac{E_m}{E_1}\frac{\rho_c}{\rho_m}\beta_m \qquad \beta_2 = (1+\nu_m)\frac{\rho_c}{\rho_m}\beta_m - \beta_1\nu_{12}$$

Further simplification for composites such as graphite/epoxy with high fiber-to-matrix moduli ratio (E_f/E_m) and no moisture absorption by fibers leads to

$$\beta_1 = 0, \text{ and} \qquad \beta_2 = (1+\nu_m)\frac{\rho_c}{\rho_m}\beta_m$$

Experimental determination: A specimen is placed in water and the moisture expansion strain is measured in the longitudinal and transverse directions. Because moisture attacks strain gage adhesives, micrometers are used to find the swelling strains.

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Homework of chapter 3

3.1 The weight fraction of aramid in a aramid/epoxy composite is 0.8. If the specific gravity of aramid and epoxy is 1.4 and 1.2 (A.K. Kaw, Mechanics of Composite Materials, 2nd ed., Taylor & Francis, 2006), respectively, find the following

1. Fiber and matrix volume fractions
2. Density of the composite

3.2 Give a unidirectional graphite/epoxy lamina with 60% fiber volume fraction. The properties of graphite and epoxy from Table 3.1 and 3.2 (A.K. Kaw. Mechanics of Composite Materials. 2nd edition, Taylor & Francis, 2006), find the following

1. Elastic moduli of the lamina using mechanics of materials approach.
2. Elastic moduli of the lamina using the Halphin–Tsai approach. Assume that the fibers are circularly shaped and are in a square array.
3. Five strength parameters.

3.3 A hybrid lamina uses glass and graphite fibers in a matrix of epoxy for its construction. The fiber volume fractions of glass and graphite are 40 and 20%, respectively. The specific gravity of glass, graphite, and epoxy is 2.6, 1.8, and 1.2, respectively. Find the following

1. Mass fractions
2. Density of the composite

3.4 A resin hybrid lamina is made by reinforcing graphite fibers in two matrices: resin A and resin B. The fiber weight fraction is 40%; for resin A and resin B, the weight fraction is 30% each. If the specific gravity of graphite, resin A, and B is 1.2, 2.6, and 1.7, respectively, find the following

1. Fiber volume fraction
2. Density of composite